CHAPTER 3. PROOFS BY DEDUCTION

| Modus ponens: | $\alpha \rightarrow \beta$ | Modus tollens: | $\alpha \rightarrow \beta$
| | $\alpha$ | | $\neg \beta$
| | | $\beta$ | $\neg \alpha$
| $\land$ introduction: | $\alpha$ | $\land$ elimination: | $\alpha \land \beta$
| | $\beta$ | | $\alpha$ [or $\beta$]
| | $\alpha \land \beta$ | | $\alpha \land \beta$
| $\lor$ introduction: | $\alpha$ [or $\beta$] | $\lor$ elimination: | $\alpha \lor \beta$
| | | (Case analysis) | $\alpha \rightarrow \gamma$
| | $\alpha \lor \beta$ | | $\beta \rightarrow \gamma$
| | | | $\gamma$
| $\neg \neg$ introduction: | $\alpha$ | $\neg \neg$ elimination: | $\neg \neg \alpha$
| | | $\neg \neg \alpha$ | $\alpha$
| $\leftrightarrow$ introduction: | $\alpha \rightarrow \beta$ | $\leftrightarrow$ elimination: | $\alpha \leftrightarrow \beta$
| | $\beta \rightarrow \alpha$ | | $\alpha \leftrightarrow \beta$
| | $\alpha \leftrightarrow \beta$ | | $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$
| Contradiction: | $\alpha$ | Tautology: | (when $\alpha \equiv \text{TRUE}$)
| | $\neg \alpha$ | | $\alpha$
| | | FALSE

Figure 3.1: Rules of Inference

In the next section we will explain how to construct proofs using these. A curious fact will emerge. We can use some rules of inference to prove that other rules are sound. In fact we only need to be “given” two rules (several choices of two work) to prove all the rest are sound. That is only of theoretic interest. What is more important here is that we have a small but powerful set of rules, so that the reader will regard the proofs as “natural,” after all the technique is called natural deduction. The list in the Figure 3.1 is arbitrary but fairly standard between textbooks.

A couple of remarks are indicated. First the Contradiction rule seems to be inconsistent with the overarching scheme of starting with true statements (axioms) and thereafter generating only true statements. That is correct. However when we introduce assumptions later, the utility of this rule will become apparent. Second the Tautology rule simply states that if there is a proposition that the reader agrees is true then it can be included. Generally this will be