Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$. 
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$. Let $p$ to be the number of states in $M$. For $w = w_1, \ldots, w_n$, let $q_0, \ldots, q_n$ be the sequence of states that lead from start to accept on string $w$. 
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Since \( L \) is regular, \( \exists \) DFA \( M \) that recognizes \( L \).

Let \( p \) to be the number of states in \( M \). For \( w = w_1, \ldots, w_n \), let \( q_0, \ldots, q_n \) be the sequence of states that lead from start to accept on string \( w \).

Because \( n > p \), some state must repeat in this sequence. Let \( q^* \) be the first to repeat.
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$.
Let $p$ to be the number of states in $M$. For $w = w_1, \ldots, w_n$, let $q_0, \ldots, q_n$ be the sequence of states that lead from start to accept on string $w$.
Because $n > p$, some state must repeat in this sequence. Let $q^*$ be the first to repeat.
Let $s$ be the index of first appearance of $q^*$, $t$ be the index of the first repetition of $q^*$. 
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$. Let $p$ to be the number of states in $M$. For $w = w_1, \ldots, w_n$, let $q_0, \ldots, q_n$ be the sequence of states that lead from start to accept on string $w$. Because $n > p$, some state must repeat in this sequence. Let $q^*$ be the first to repeat. Let $s$ be the index of first appearance of $q^*$, $t$ be the index of the first repetition of $q^*$. Let $x = w_1 \cdots w_s$, $y = w_{s+1} \cdots w_t$, and $z = w_{t+1} \cdots w_n$. 
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$. Let $p$ to be the number of states in $M$. For $w = w_1, \ldots, w_n$, let $q_0, \ldots, q_n$ be the sequence of states that lead from start to accept on string $w$. Because $n > p$, some state must repeat in this sequence. Let $q^*$ be the first to repeat. Let $s$ be the index of first appearance of $q^*$, $t$ be the index of the first repetition of $q^*$. Let $x = w_1 \cdots w_s$, $y = w_{s+1} \cdots w_t$, and $z = w_{t+1} \cdots w_n$. 

![Diagram of DFA with states labeled and transitions indicated.](image-url)
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \(|w| \geq p\), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \(|y| > 0\)
3. \(|xy| \leq p\)

Since \( L \) is regular, \( \exists \) DFA \( M \) that recognizes \( L \).
Let \( p \) to be the number of states in \( M \). For \( w = w_1, \ldots, w_n \), let \( q_0, \ldots, q_n \) be the sequence of states that lead from start to accept on string \( w \).
Because \( n > p \), some state must repeat in this sequence. Let \( q^* \) be the first to repeat.
Let \( s \) be the index of first appearance of \( q^* \), \( t \) be the index of the first repetition of \( q^* \).
Let \( x = w_1 \cdots w_s \), \( y = w_{s+1} \cdots w_t \), and \( z = w_{t+1} \cdots w_n \).
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, x y^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Since $L$ is regular, $\exists$ DFA $M$ that recognizes $L$. Let $p$ to be the number of states in $M$. For $w = w_1, \ldots, w_n$, let $q_0, \ldots, q_n$ be the sequence of states that lead from start to accept on string $w$. Because $n > p$, some state must repeat in this sequence. Let $q^*$ be the first to repeat. Let $s$ be the index of first appearance of $q^*$, $t$ be the index of the first repetition of $q^*$. Let $x = w_1 \cdots w_s$, $y = w_{s+1} \cdots w_t$, and $z = w_{t+1} \cdots w_n$. To see that property 3 is satisfied: recall, $t$ is the first repetition. If $t > p$, we must have more than $p$ states.
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0$, $xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L = \{a^n b^n | n \geq 0\}$ is not regular.
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L = \{a^n b^n | n \geq 0\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satisfied
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L = \{a^n b^n | n \geq 0 \}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satsified

We want to show:

$\neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satsified)
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L = \{a^n b^n | n \geq 0\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all } 3 \text{ properties are satisfied}$

We want to show:

$\neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all } 3 \text{ properties are satisfied})$

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all } 3 \text{ properties are satisfied}$
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L = \{a^nb^n | n \geq 0\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}$

We want to show:

$\neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied})$

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied}$

Assume $p \geq 1$ (towards $\forall$ introduction)
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L = \{a^n b^n | n \geq 0\} \) is not regular.

Our lemma says:
\[ \exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied} \]
We want to show:
\[ \neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}) \]
\[ \equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied} \]
Assume \( p \geq 1 \) (towards \( \forall \) introduction)
Let \( w = a^p b^p \)
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L = \{ a^n b^n \mid n \geq 0 \} \) is not regular.

Our lemma says:
\[ \exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified} \]

We want to show:
\[ \neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified}) \]
\[ \equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satsified} \]

Assume \( p \geq 1 \) (towards \( \forall \) introduction)
Let \( w = a^p b^p \)
Claim: \( \forall x, y, z, w \) violates (at least) one of the 3 properties.
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L = \{a^nb^n | n \geq 0\}$ is not regular.

Our lemma says:
$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified}$

We want to show:
$\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified})$
$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satsified}$

Assume $p \geq 1$ (towards $\forall$ introduction)
Let $w = a^pb^p$

Claim: $\forall x, y, z$, $w$ violates (at least) one of the 3 properties.
If property 3 is violated, we’re done.
Suppose property 3 is not violated.
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L = \{a^n b^n | n \geq 0\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}$

We want to show:

$\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied})$

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied}$

Assume $p \geq 1$ (towards $\forall$ introduction)

Let $w = a^p b^p$

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.

If property 3 is violated, we’re done.
Suppose property 3 is not violated.
If property 2 is violated, we’re done.
Suppose property 2 is not violated.
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L = \{a^nb^n | n \geq 0\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satisfied

We want to show:

$\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satisfied)

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z$ s.t. NOT all 3 properties are satisfied

Assume $p \geq 1$ (towards $\forall$ introduction)

Let $w = a^pb^p$

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.

If property 3 is violated, we're done.

Suppose property 3 is not violated.

If property 2 is violated, we’re done.

Suppose property 2 is not violated.

Then $y \in \{aa^j \}^{P-1}_{j=0}$. $\forall i \geq 0 : xy^i z \notin L$. 
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{ (, ) \}$ is not regular.
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{\(', \)'\}$ is not regular.

Examples of strings in the language: $(((())())()$ and $()(()())$
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L$ of “balanced parenthesis” over $\Sigma = \{\(',\)'\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}$
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^i z \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{\('', '）'\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}$

We want to show:

$\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied})$
Limitations on regular languages

Pumping Lemma
If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^i z \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem
The language \( L \) of “balanced parenthesis” over \( \Sigma = \{‘(’, ‘)’\} \) is not regular.

Our lemma says:
\[ \exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified} \]
We want to show:
\[ \neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified}) \]
\[ \equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satsified} \]
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L \) of “balanced parenthesis” over \( \Sigma = \{('(', ')')\} \) is not regular.

Our lemma says:
\[ \exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied} \]

We want to show:
\[ \neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}) \]
\[ \equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied} \]

Assume \( p \geq 1 \) (towards \( \forall \) introduction)
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{\(',\)'\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satisfied

We want to show:

$\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z$ s.t. all 3 properties are satisfied)$

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z$ s.t. NOT all 3 properties are satisfied

Assume $p \geq 1$ (towards $\forall$ introduction)

Let $w = (\underbrace{((\cdots(\underbrace{((())\cdots}))\cdots)))}_{p \text{ times}}$

$p \text{ times}$
Limitations on regular languages

### Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

### Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{('(', ')')\}$ is not regular.

Our lemma says:

$\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}$

We want to show:

$\neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied})$

$\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied}$

Assume $p \geq 1$ (towards $\forall$ introduction)

Let $w = (((\cdots (((()))\cdots)))

$p$ times $p$ times

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^i z \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L \) of “balanced parenthesis” over \( \Sigma = \{‘(’, ‘)’\} \) is not regular.

Our lemma says:
\[
\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified}
\]
We want to show:
\[
\neg (\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satsified})
\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satsified}
\]
Assume \( p \geq 1 \) (towards \( \forall \) introduction)
Let \( w = (((\cdots (((()))\cdots )))\))

\[\begin{array}{c}
p \times \\
\text{p times}
\end{array}\]

Claim: \( \forall x, y, z, w \) violates (at least) one of the 3 properties.
Assume properties 2 and 3 are satisfied.
Limitations on regular languages

**Pumping Lemma**

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

**Theorem**

The language \( L \) of “balanced parenthesis” over \( \Sigma = \{‘(’, ‘)’\} \) is not regular.

Our lemma says:
\[
\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied}
\]

We want to show:
\[
\neg(\exists p : \forall w \in L : |w| \geq p : \exists x, y, z \text{ s.t. all 3 properties are satisfied})
\]

\[
\equiv \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied}
\]

Assume \( p \geq 1 \) (towards \( \forall \) introduction)

Let \( w = (((...(((())))...))) \)

\( p \) times \( p \) times

Claim: \( \forall x, y, z, w \) violates (at least) one of the 3 properties.

Assume properties 2 and 3 are satisfied.

Then \( y \in \{((j)}_{j=0}^{p-1} \cdot \forall i \geq 0 : xy^iz \notin L \).
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L \) of “balanced parenthesis” over \( \Sigma = \{\(',\,')\} \) is not regular.

We want to show:
\( \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied} \)
Assume \( p \geq 1 \) (towards \( \forall \) introduction)
Wrong answer:
Let \( w = (())(\cdots(())) \)
\( p \) times
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{('(', ')')\}$ is not regular.

We want to show:
$\forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied}$

Assume $p \geq 1$ (towards $\forall$ introduction)

Wrong answer:

Let $w = \underbrace{(()) \cdots ()()}_{p \text{ times}}$

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.
Limitations on regular languages

**Pumping Lemma**

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

**Theorem**

The language $L$ of “balanced parenthesis” over $\Sigma = \{\('',\)\}$ is not regular.

We want to show:
$\forall p : \exists w \in L : |w| \geq p : \forall x, y, z$ s.t. NOT all 3 properties are satisfied

Assume $p \geq 1$ (towards $\forall$ introduction)

Wrong answer:
Let $w = \underbrace{()()\cdots()()\cdots()}_{p \text{ times}}$

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.
Assume properties 2 and 3 are satisfied.
Limitations on regular languages

Pumping Lemma

If $L$ is a regular language, then there exists an integer $p \geq 1$ such that for any $w \in L$ with $|w| \geq p$, $w$ can be divided into 3 strings, $w = xyz$ such that:

1. $\forall i \geq 0, xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

Theorem

The language $L$ of “balanced parenthesis” over $\Sigma = \{\text{', '}\}$ is not regular.

We want to show:
$\forall p : \exists w \in L : |w| \geq p : \forall x, y, z$ s.t. NOT all 3 properties are satisfied
Assume $p \geq 1$ (towards $\forall$ introduction)
Wrong answer:
Let $w = (())() \cdots ()()$

$p$ times

Claim: $\forall x, y, z, w$ violates (at least) one of the 3 properties.
Assume properties 2 and 3 are satisfied.
If $y = ()$, $xy^iz \in L$. 
Limitations on regular languages

Pumping Lemma

If \( L \) is a regular language, then there exists an integer \( p \geq 1 \) such that for any \( w \in L \) with \( |w| \geq p \), \( w \) can be divided into 3 strings, \( w = xyz \) such that:

1. \( \forall i \geq 0, xy^i z \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Theorem

The language \( L \) of “balanced parenthesis” over \( \Sigma = \{‘(’, ‘)’\} \) is not regular.

We want to show:
\( \forall p : \exists w \in L : |w| \geq p : \forall x, y, z \text{ s.t. NOT all 3 properties are satisfied} \)

Assume \( p \geq 1 \) (towards \( \forall \) introduction)

Wrong answer:

Let \( w = (())(()) \cdots (()) \) 
\( p \) times

Claim: \( \forall x, y, z, w \) violates (at least) one of the 3 properties.

Assume properties 2 and 3 are satisfied.

If \( y = () \), \( xy^i z \in L \).
\( \neg \forall x, y, z : \forall i \geq 0 : xy^i z \notin L. \)