Advanced Queueing Theory

M/G/1 Queueing Systems

- Service times have a general distribution.
- Other assumptions of M/M/1 are retained.

Implications:

- We can no longer rely on the the memoryless property of service times.
- If we were to use the state transition diagram approach, then each state must contain $(N(t), A(t))$, where $N(t)$ is the number of customers at time t and $A(t)$ represents how long the customer in the server has been served up to time t.
- Can you explain why we don't need $A(t)$ in M/M/* ?

Notations

- W_i --- waiting time in queue of the *i*th customer
- R_i --- residual service time of the currently served customer upon the arrival of the *i*th customer
- X_i --- service time of the *i*th customer
- N_i --- number of customers found waiting in queue by the *i*th customer upon his arrival

Analysis

By definition,

$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$\Rightarrow E\{W_i\} = E\{R_i\} + E\Big\{\sum_{j=i-N_i}^{i-1} X_j\Big\} = E\{R_i\} + E\{X\}E\{N_i\}$$

$$\Rightarrow W = R + \frac{1}{\mu} N_Q, \quad \text{(by taking the limit } i \to \infty)$$

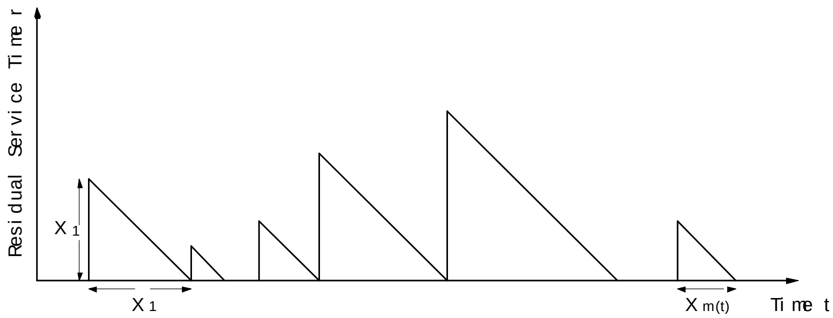
$$\Rightarrow W = R + \frac{1}{\mu}(\lambda W), \quad \text{(by Little's Theorem)}$$

$$\Rightarrow W = \frac{R}{1-\rho}$$

where R is the average residual service time.

Residual Times

- $r(t)$ --- the residual service time at time t
- $m(t)$ --- the number of service completions up to time t.



Let us compute the time average of $r(t)$ in the interval $(0,t)$:

$$\frac{1}{t}\int_0^t r(s)ds = \frac{1}{t}\sum_{i=1}^{m(t)}\frac{1}{2}X_i^2 = \frac{1}{2}\left(\frac{m(t)}{t}\right)\left(\frac{\sum_{i=1}^{m(t)} X_i^2}{m(t)}\right)$$

$$\Rightarrow \lim_{t\to\infty}\frac{1}{t}\int_0^t r(s)ds = \frac{1}{2}\left(\lim_{t\to\infty}\frac{m(t)}{t}\right)\left(\lim_{t\to\infty}\frac{\sum_{i=1}^{m(t)} X_i^2}{m(t)}\right)$$

$$\Rightarrow R = \frac{1}{2}\lambda\overline{X^2}, \qquad \text{where } \overline{X^2} = E[X^2]$$

Recalling that $W = R/(1-\rho)$, we now have the Pollaczek-Khinchin (P-K) formula:

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)}$$

Average time in system ($\overline{X} = 1/\mu$ is the average service time):

$$T = \overline{X} + W = \overline{X} + \frac{\lambda \overline{X^2}}{2(1-\rho)}$$

Average number of customers in queue:

$$N_Q = \lambda W = \overline{X} + \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}$$

Average number of customers in system:

$$N = \lambda T = \rho + \frac{\lambda^2 \overline{X^2}}{2(1-\rho)}$$

G/M/1 Queueing Systems

- Interarrival times form a general distribution with pdf $g(t)$.
- All other M/M/1 assumptions are retained.
- As in the case of M/G/1 queues, we cannot summarize the state of the entire system in a single number, the number of customers in system.
- Instead, we will focus on the behavior of the system at some "special moments" when the state of the system can be summarized in one number.

State Probability Revisited

- For M/M/1 queues, we solved P_i , the probability for the system to be in state ***i***.
- If you think carefully, the state probability changes over time.
  - consider a barbershop whose $\lambda = 4$ customers per hour and whose $\mu = 5$ customers per hour
  - according to our formula, $P_3 = 0.8^3 \times 0.2 = 0.1024$
  - however, what is the chance of seeing 3 customers in the shop in the first 1 second ?
  - must be very very small ! (certainly less than 10%)
- What does P_i really mean ?

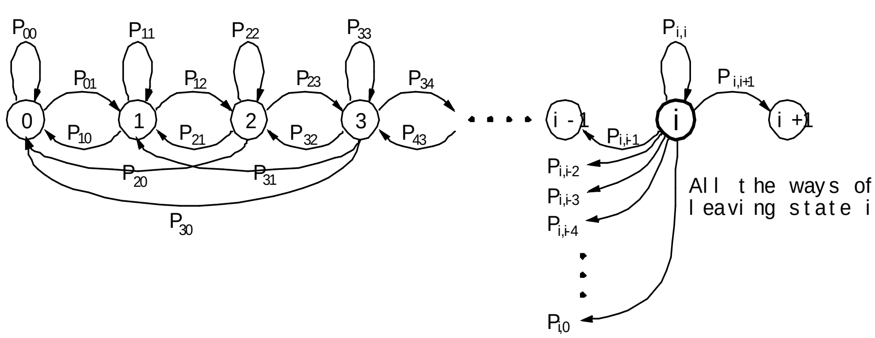
- Let $P_i(t)$ be the probability of having *i* customers during the interval [0,t], that is

$$P_i(t) = \frac{\text{the portion in } [0, t] \text{ when the system has } i \text{ customers}}{\text{the length of the interval, } t}$$

- P_i is defined as $P_i = \lim_{t \to \infty} P_i(t)$
- That is, P_i is the average probability of state *i* over an indefinitely long period of time, taking all time points into account.
- It turns out that P_i is difficult to obtain with G/M/1 queues.

- ❑ Rather than finding the probability over all time points, we shall content ourselves with the system's behavior only at the moments of customer arrivals.
- ❑ Precisely, let π_i be the probability that an arriving customer sees *i* customers in the system.
- ❑ Can you see that the knowledge of π_i is rather limited ?
  – with P_i , we know the probability of state *i* at all times, as long as the system has been running long enough
  – with π_i , we know the probability of state *i* only at the moments of customer arrivals
- ❑ On the positive side, the system CAN be summarized in a single number at such moments. Why ?

State Transition Diagram



- ❑ A transition represents a customer arrival.
- ❑ $P_{i,j}$ represents the probability of moving from *i* to *j* upon a new arrival.

Reading the Diagram

- How do we enter state *i* from *i* ?
  - the system had *i* customers when the previous customer arrived
  - it has *i* customers when the next customer arrives
  - this means that exactly one customer has been served and left the system between the two arrivals
- In general, a transition from *i* to *i* +1-*j* means *j* customers have been served between two consecutive arrivals.

- It can be shown that (see Appendix):

$$\pi_k = (1-\beta)\,\beta^k$$

where

$$\beta = \int_0^\infty e^{-ut(1-\beta)}\,dG(t)$$

- We can obtain the value of β through numeric methods.

- Notice that, when an arrival sees *k* customers in system, then he spends *k*+1 service periods in the system, implying

$$T = \sum_{k=0}^{\infty} \frac{k+1}{\mu} \pi_k = \sum_{k=0}^{\infty} \frac{k+1}{\mu} (1-\beta)\, \beta^k = \frac{1}{\mu(1-\beta)}$$

- Finally, the average number of customers in the system is

$$N = \lambda T = \frac{\lambda}{\mu(1-\beta)}$$

- Amazingly, the above *T* and *N* formulas are unconditional, that is, they are valid at all times, not just the moments of arrivals.

Appendix

- By definition,

$$P_{i,i+1-j} = \int_0^{\infty} e^{-ut} \frac{(ut)^j}{j!} g(t)dt, \qquad j = 0,1,\ldots,i$$

$$P_{i,0} = 1 - \sum_{j=0}^{i} P_{i,i+1-j}$$

- By the nature of π, we have

$$\pi_k = \sum_i \pi_i P_{ik}, \qquad k \geq 0$$

and

$$\sum_k \pi_k = 1$$

- The above can be reduced to

$$\pi_k = \sum_{i=k-1}^{\infty} \pi_i \int_0^{\infty} e^{-ut} \frac{(ut)^{i+1-k}}{(i+1-k)!} dG(t), \qquad k \ge 1$$

and

$$\sum_0^{\infty} \pi_k = 1$$

- Let us try a solution of the form $\pi_k = c\,\beta^k$. That is,

$$c\,\beta^k = \sum_{i=k-1}^{\infty} c\,\beta^i \int_0^{\infty} e^{-ut} \frac{(ut)^{i+1-k}}{(i+1-k)!} dG(t)$$

$$= c \int_0^{\infty} e^{-\mu t} \beta^{k-1} \sum_{i=k-1}^{\infty} \frac{(\beta \mu t)^{i+1-k}}{(i+1-k)!} dG(t)$$

- However, $\sum_{i=k-1}^{\infty} \frac{(\beta \mu t)^{i+1-k}}{(i+1-k)!} = \sum_{j=0}^{\infty} \frac{(\beta \mu t)^j}{j!} = e^{\beta \mu t}$

$$\Rightarrow \beta^k = \beta^{k-1} \int_0^{\infty} e^{-ut(1-\beta)} dG(t)$$

$$\Rightarrow \beta = \int_0^{\infty} e^{-ut(1-\beta)} dG(t)$$

- We can obtain the value of β through numeric methods.
- Since $\sum_0^{\infty} \pi_k = \sum_0^{\infty} c\,\beta^k = 1$, we have $c = 1 - \beta$.
- It follows that the conditional probability

$$\pi_k = (1-\beta)\,\beta^k$$