CS483 Analysis of Algorithms
Lecture 09 – Linear Programming 01 *

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Introduction

Linear Programming

- Similar to dynamic programming, “programming” here means *optimization*
- Linear programming (LP) problems are optimization problems whose **objective** and **constraints** are all *linear* (i.e., exponents of all variables are 1)
- Many real-life problems can be expressed as LP problems
  - Example: Profit maximization
    - You are selling two kinds of chocolates: Pyramide and Pyramide Nuit
    - You make \$1 profit by selling one box of Pyramide and \$6 profit by selling one box of Pyramide Nuit
    - Your factory can only make 200 and 300 boxes of Pyramide and Nuit, resp., per day
    - Your worker can only produce 400 boxes per day.
    - You want to maximize your profit
  - How many boxes of Pyramide and Pyramide Nuit do you make to maximize your profit?

Example: Profit maximization

- Let x_1 and x_2 be the number of boxes we want to produce for Pyramide and Pyramide Nuit.
- **Objective Function**:
- **Constraints**:

  1.

  2.

  3.

  4.

- A LP problem can have **zero**, **one**, or **infinity** optimal solutions

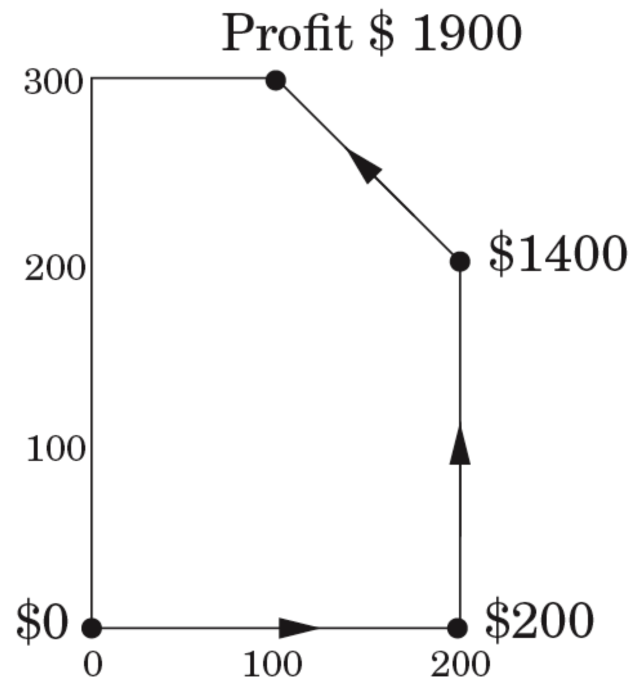
  1. $x > 5, x \leq 3$
  2. $\max\{x_1 + x_2\}, x_1, x_2 > 0$

Geometric Interpretations of LP problems

- Each linear constraint can be represented as a **halfspace**
- A set of feasible solutions of a LP problem forms a **convex** set
- The objective function can be represented as a **hyperplane**
- When there is a unique solution, this solution must be a vertex of the convex set formed by the constraints
- Example: **maximize** $x_1 + 6x_2$

$$\begin{array}{rcl} x_1 & \leq & 200 \\ x_2 & \leq & 300 \\ x_1 + x_2 & \leq & 400 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

- LPs can be solved by the *simplex method* (named one of the top ten best algorithms in 20th century)
- Closely related to *hill-climbing* by jumping from one vertex to an adjacent vertex



- Simplex is a type of "iterative improvement" method
- We will cover simplex in the next lecture (for now we assume we have a simplex package that solves our problems).

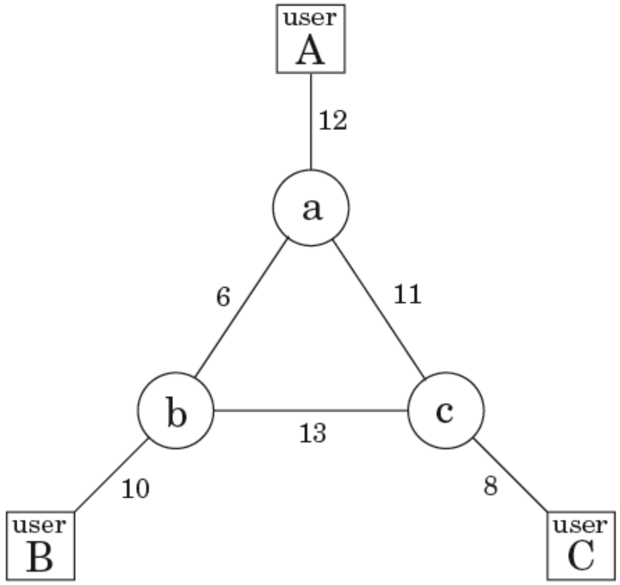
Example: Production Planning

- We have a company making hand-made carpets and today is Jan/1st.
  - We now have 30 employees and each of them makes 20 carpets and get $2000 per month.
  - Each employee gets paid 80% more by working overtime but can only put in at most 30% overtime.
  - We can hire and fire employee. Hiring costs $320 and firing costs $400 per worker.
  - Storing surplus will cost $8 per carpet per month.
  - We do not have surplus now and we must end the year without surplus.
  - The demand for all months are $d_1, d_2, \ldots, d_12$
- How do we minimize our total cost?

Example: Production Planning

Example: Bandwidth Allocation

- Our company now is a network services provider
  - The network has 3 nodes: A, B, C
  - Connection $A - B$ pays \$3 per unit of bandwidth
  - Connection $B - C$ pays \$2 per unit of bandwidth
  - Connection $A - C$ pays \$4 per unit of bandwidth
  - Each connection requires at least two units of bandwidth
  - Each connection can be routed in two ways: long and short routes
  - Bandwidths of the network are shown below



- How do we route these connections to maximize our network's revenue?

Example: Bandwidth Allocation

LP variants and Standard form

- Variants
  1. Objective functions: maximization and minimization
  2. Constraints: equation or/and inequalities
  3. Restrictions: variables are often restricted to be non-negative
- Standard form
  1. Objective functions: minimization
  2. Constraints: equation
  3. Restrictions: variables are all non-negative
- Reduction to standard form

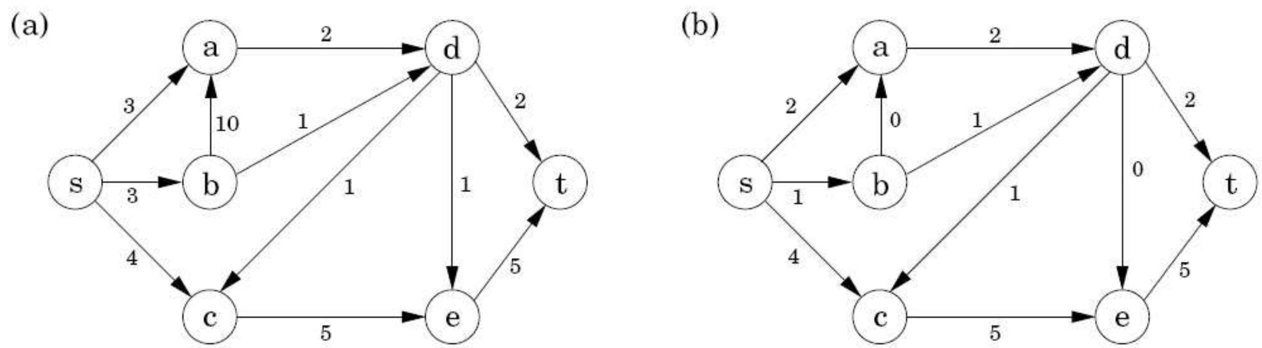
maximize $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1 &\geq 0 \end{aligned}$$

Flows in networks

Maximum-flow problem

- Assuming that you are working for an oil company and the company owns a network of pipe lines along which oil can be sent, you are asked to find out the maximum capacity of oil can be sent from a city s to another city t over the network.



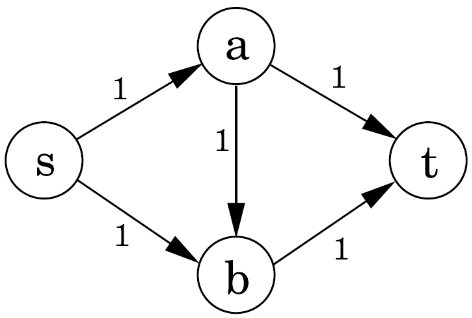
- **Maximum-flow problem**: Given a weighted direct graph $G = \{V, E\}$, whose edge weight indicates the maximum capacity of an edge, find the maximum flow from a vertex s (source) and to another vertex t (sink) so that the following requirements are satisfied.
  - The flow f_e on edge e must be $0 \le f_e \le c_e$
  - Flow is conserved, i.e., $\sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{vw}$

LP and Maximum-flow problem

- **Variables**:
- **Objective**:
- **Constraints**:

Maximum-flow problem

- Iterative improvement
  - Start with 0 capacity
  - **Repeat**: Find a path from s to t, and increase the flow along this path as much as possible
- Example:

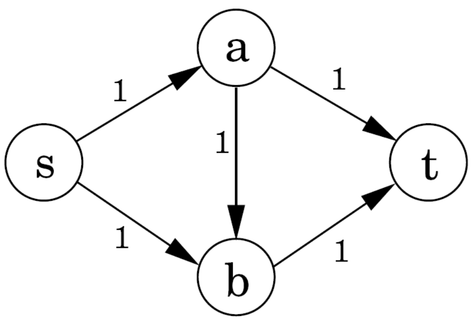


Residual graph

- To make the algorithm work: We allow path to **cancel existing flow**
- Residual graph G^f, whose edge weight indicate the remaining capacity of an edge. Two types of edge weights are available in G^f:

1. $c_{uv} - f_{uv}$, if (u, v) is an edge of G and $f_{uv} < c_{uv}$
2. f_{vu}, if (u, v) is an edge of G and $f_{uv} > 0$

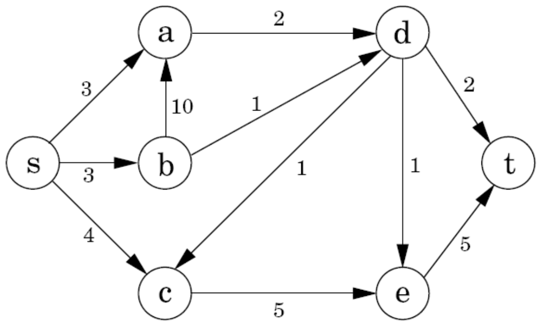
- Example:



Example

☐ Example

Flow



Residual graph G^f

Example

Flow

Residual graph G^f

- Time complexity:

Minimum Cut

- **Graph cut**: (s, t)-cut is the removal of a set of edges so that a connected component splits s and t into two connected components

- The total capacity (edge weights) of a cut is an upper-bound of the capacity flow from one component to the other component

- Theorem: **Maximum-flow Minimum cut**: The maximum flow of a graph from s to t equals to the capacity of the smallest (s, t)-cut

- Question: How to compute the minimum cut of a given graph?

Maximum Bipartite Matching

- ☐ Given n men and n women, we add an edge between a man and a woman if they like each other. Can you find a *perfect matching*?

- ☐ A graph is **bipartite** if you can split the vertices to two groups such that there is no edge connecting vertices in the same group

A bipartite graph

Not a bipartite graph

Maximum Bipartite Matching

- Solving maximum bipartite matching problem:

Stable Matching

- Let's make the problem more realistic: Given n men and n women, every man (woman) will rank all women (men).
- We say a set of marriages (matching) is unstable if there are two pairs (m, w) and (m', w') with the following properties:

  1. m prefers w' to w
  2. w' prefers m to m'

- Example 1 (m,m',w,w'):

  1. m prefers w to w'
  2. m' prefers w to w'
  3. w prefers m to m'
  4. w' prefers m to m'

- Example 2 (m,m',w,w'):

  1. m prefers w to w'
  2. m' prefers w' to w
  3. w prefers m' to m
  4. w' prefers m to m'

- Given n men and n women and a list of preferences, can you find a stable marriage for them?

Stable Matching

- Ideas:
  - The idea is to have the pair (m, w) enter a state called "engagement" before marriage
  - A *free* (not engaged) man m can *propose* to a women w, there will be two possibilities:

    1. w rejects m (when w prefers her fiancee)
    2. w and m are engaged (when w is free or w prefers m)

  - A man can only propose to a woman once

Stable Matching

- Algorithm

Algorithm 0.1: STABLEMATCHING(n)

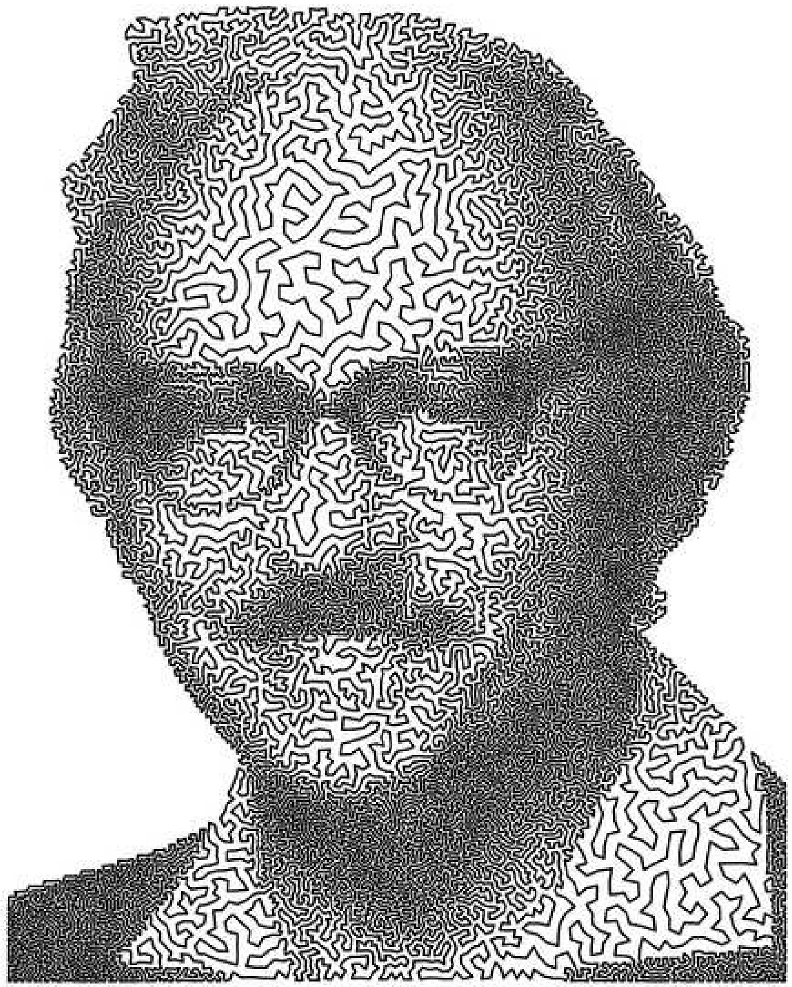
while there are free men
do
  pick a free man m
  Let w be the woman with the highest ranking, to whom m has not yet proposed
  if w is free
    then $w$and m are engaged
    else
      if w prefers m'
        then m is still free
        else
          w and m are engaged
          m' is now free
Each engaged couple are now married

- What is the time complexity?

Stable Matching

- Properties
  - A woman remain engaged after she was proposed first time. Her fiancee gets better and better.
  - A man can become free after engagement (his fiancee left him). His fiancee get worse and worse.
  - This algorithm is biased to man: the matching is always a **man-optimal** matching
- Is the algorithm correct?

Simplex



TSPortrait of Dantzig by Robert Bosch. George Dantzig (1914-2005) was the father of linear programming and the inventor of the Simplex Method.

Simplex Algorithm

- Simplex algorithm is an iterative improvement method
  - starting with a vertex v of the convex set (of feasible solutions)
  - find another vertex v' adjacent to v with a higher objective value
  - $v = v'$, until no better adjacent vertex
- Example:**maximize** $x_1 + 6x_2$

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

- Some more geometry
  - A vertex is formed by intersecting n constraints (for a problem with n variables)
  - Two adjacent vertices will share $n-1$ constraints (and one different constraint)

Simplex Algorithm

- For a the simplex algorithm, we need to:
  - find an initial solution
  - update the current solution
- In some cases, our initial point is simple, i.e., $(0, 0, \cdots, 0)$, which gives us many advantages:
  1. This vertex is the intersection of $x_i \geq 0$ constraints
  2. When all coefficients in the objective function are **negative**, our initial solution is optimal
  3. To pick an adjacent vertex, we simply pick a variable x_i whose coefficient in the objective function is positive and try to maximize x_i
- Example: **maximize** $x_1 + 6x_2$

$$\begin{array}{rcl} x_1 & \leq & 200 \\ x_2 & \leq & 300 \\ x_1 + x_2 & \leq & 400 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

Simplex Algorithm

- Now, what do we do when our current solution is not at $(0, 0, \cdots, 0)$ anymore?
- Well, we transform our problem so the current solution is at $(0, 0, \cdots, 0)$
- **Transform** coordinate system:
  - Note that coordinates are defined as distances to the constraints
  - After we move to an adjacent vertex, **one** constraint is changed
  - Therefore, the coordinate defined by the new constraint needs to be updated
  - The distance from a point to a hyper-plane $a_i x = b_i$ is simply $b_i - a_i x$
- Example: **maximize** $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- Let’s finish the example

Simplex Algorithm

- What if $(0, 0, \cdots, 0)$ is not a feasible vertex? How do we start the process?
- We can modify the original LP problem by adding m **artificial** variables z_i, where m is the number of constraints. Now our new LP problem becomes:
  - $z_0 \geq 0$, $z_1 \geq 0$, ... $z_{m-1} \geq 0$
  - Add z_i to the left size of the i-th constraint
  - minimize $z_0 + z_1 + \cdots + z_{m-1}$
- First the initial vertex of the modified LP is easy to obtain: $(x_1 = 0, x_2 = 0, \cdots, x_{n-1} = 0, z_0 = b_0, z_1 = b_1, \cdots, z_{m-1} = b_{m-1})$
- Once we have the initial vertex, we can use the Simplex algorithm to solve the modified LP problem
- Now, if we have $z_0 + z_1 + \cdots + z_{m-1} = 0$, we have an initial solution to solve the original LP problem
- If $z_0 + z_1 + \cdots + z_{m-1} \neq 0$, the original LP will not have a feasible solution