CS583 Lecture 11

Linear Programming

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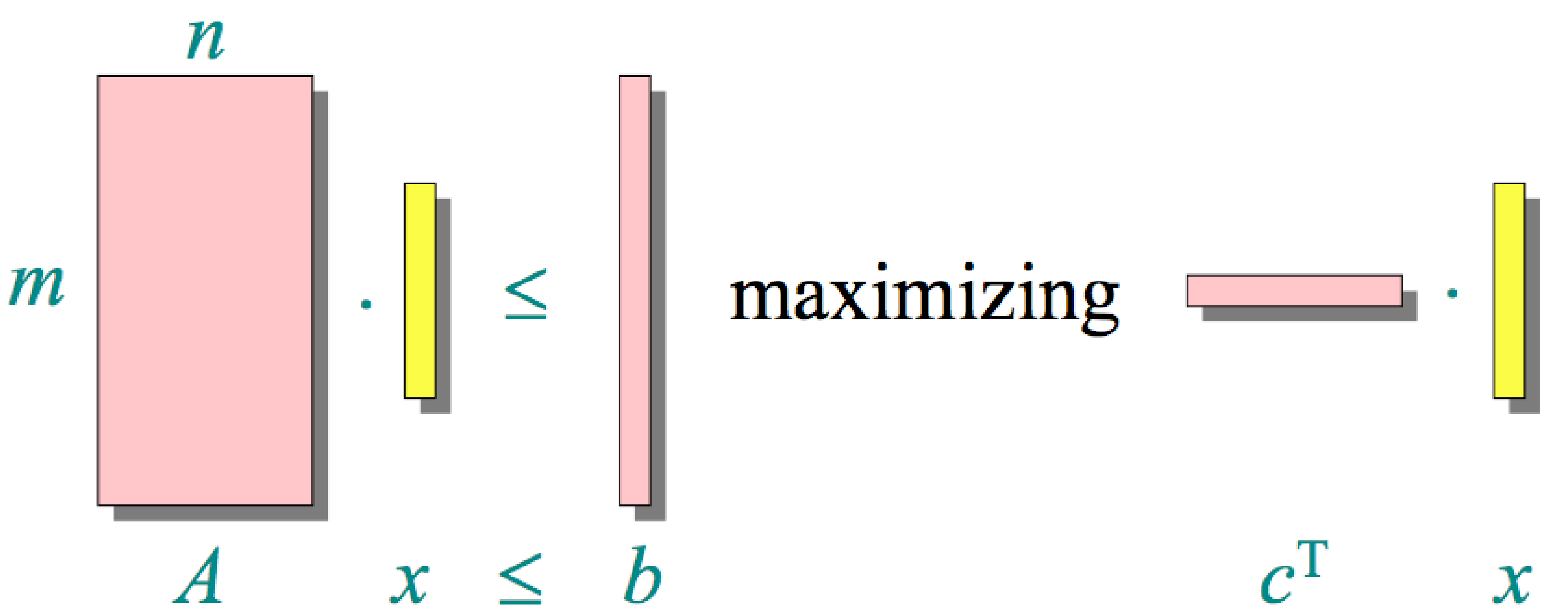
Linear Programming

- A linear programming problem usually looks like this:

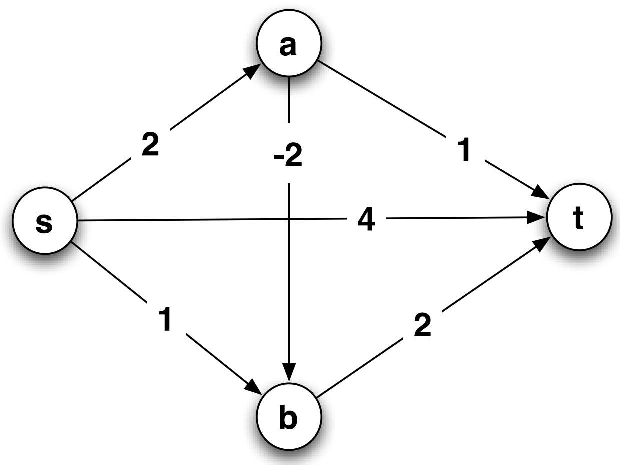


Linear Programming

- Methods that solve linear programming problems
  - Simplex methods (1947)
    - exponential worst case time
    - very fast in practice
  - ellipsoid algorithm (1979)
    - polynomial worst case time
    - slow in practice
  - Interior-point methods (1984)
    - polynomial worst case time
    - competitive with simplex methods

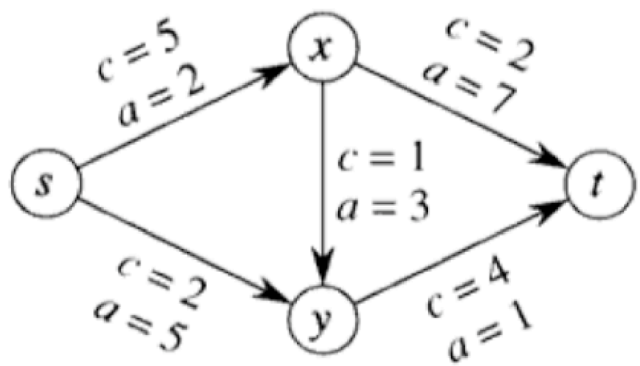
Example

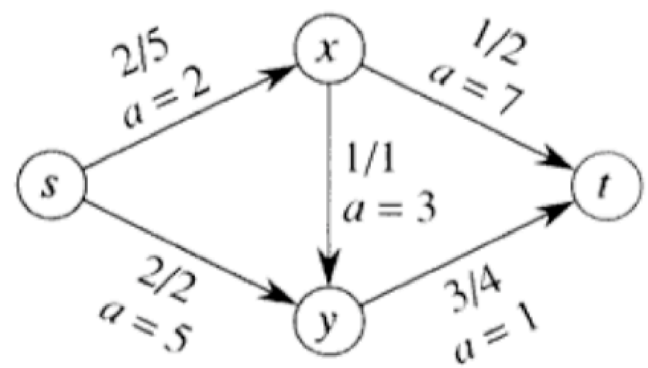
- Single-source short paths problem



Examples

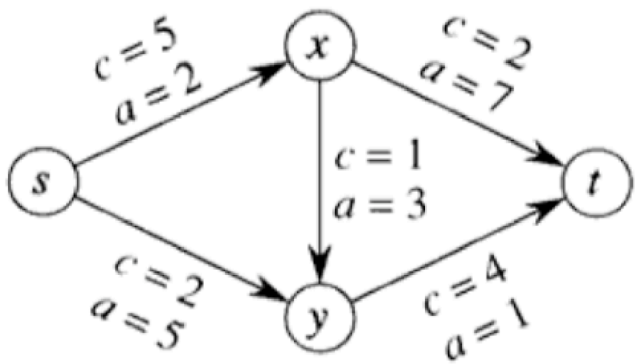
- Max flow problem

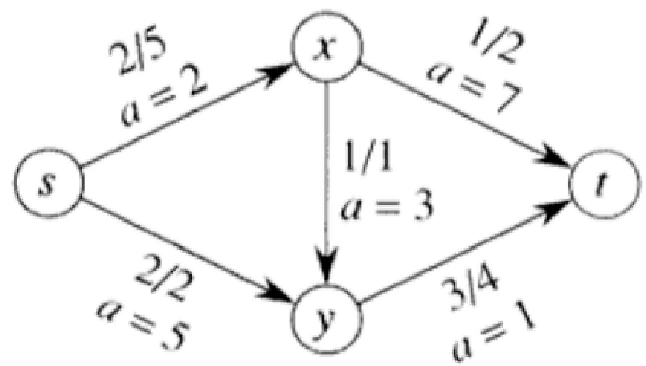




Examples

- Min-cost flow problem
  - linear programming can solve variant of problems that do not have an efficient algorithm yet





  - a(u,v): cost of each unit flow

Simplex Methods

named one of the top ten
best algorithms in 20th century

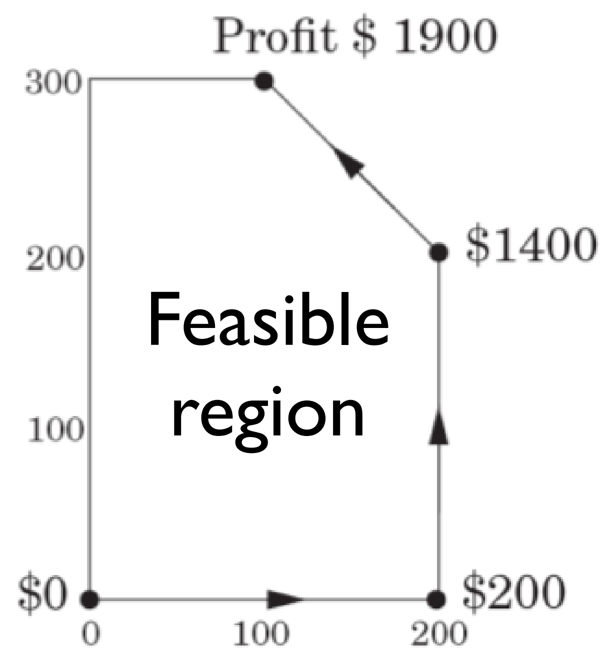
TSPortrait of Dantzig by Robert Bosch. George Dantzig (1914-2005) was the father of linear programming and the inventor of the Simplex Method.

Geometric View

Ex: Apple sells two types of ipod

Example: **maximize** $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



Optimal solution can always be found at a vertex

Simplex Methods

- Simplex is a type of "iterative improvement" method
- Key idea
  - starting with a vertex v of the convex set (of feasible solutions)
  - find another vertex v' adjacent to v with a higher objective value
  - $v = v'$, until no better adjacent vertex
- Some more geometry
  - A vertex is formed by intersecting n constraints (for a problem with n variables)
  - Two adjacent vertices will share $n - 1$ constraints (and one different constraint)
- Main steps
  - find an initial solution
  - update the current solution

Simplex Methods

- In most cases, our initial point is simple
  - (0,0,....,0)
  - this is the intersection of all $x_i \geq 0$
  - when all coefficients in the objective function are negative, this solution is optimal
  - to pick an adjacent vertex, we simply pick a variable x_i
    - whose coefficient is positive
    - try to maximize x_i

Simplex Methods

Example: **maximize** $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Simplex Methods

- What do we do if our current solution is not (0,0,...,0)?
  - we **transform our problem** so that the current solution is (0,0,...,0)
- Some more geometry
  - coordinated are defined by "distance" to the constraints (axis)
  - after moving to an adjacent vertex, one constraint (axis) is changed
  - therefore the coordinate defined by the new constraint (axis) needs to be changed
  - the distance to a hyperplane $a_i x = b_i$ is simply $b_i - a_i x$

Simplex Methods

Example: **maximize** $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

geometric interpretation

Simplex Methods

- some loose ends

- ☐ What if $(0, 0, \cdots, 0)$ is not a feasible vertex? How do we start the process?
- ☐ We can modify the original LP problem by adding m **artificial** variables z_i, where m is the number of constraints. Now our new LP problem becomes:
  - $z_0 \geq 0, z_1 \geq 0$, ... $z_{m-1} \geq 0$
  - Add z_i to the left size of the i-th constraint
  - minimize $z_0 + z_1 + \cdots + z_{m-1}$
- ☐ First the initial vertex of the modified LP is easy to obtain: $(x_1 = 0, x_2 = 0, \cdots, x_{n-1} = 0, z_0 = b_0, z_1 = b_1, \cdots, z_{m-1} = b_{m-1})$
- ☐ Once we have the initial vertex, we can use the Simplex algorithm to solve the modified LP problem
- ☐ Now, if we have $z_0 + z_1 + \cdots + z_{m-1} = 0$, we have an initial solution to solve the original LP problem
- ☐ If $z_0 + z_1 + \cdots + z_{m-1} \neq 0$, the original LP will not have a feasible solution

Simplex Methods

- time complexity? n variable, m constraints

Duality

- How do we convert a primal to a dual? Let's look at our chocolate factory example:
  maximize $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

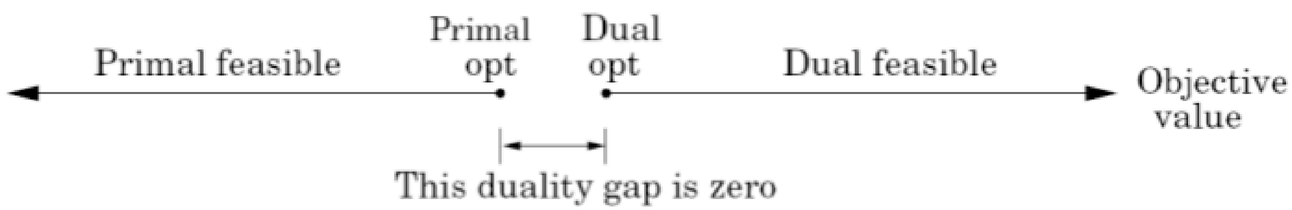
- We know that when $(x_1, x_2) = (100, 300)$, the objective function is 1900
  - Amazingly this is exact: $5 \cdot (x_2 \leq 300) + (x_1 + x_2 \leq 400)$
- Therefore, in some way, we can *verify* the optimal value by manipulating the constraints.

Duality

- How do we find the values 5 and 1 above? We introduce 3 variables $(y_1, y_2, y_3) \geq 0$ to represent these values and rewrite the objective function

Duality

- Duality is in fact a general phenomenon. It exists for all linear programming.
- **Duality Theorem**: If a linear program has a bounded optimum, then so does its dual, and two optimum values coincide.



- General Primal/Dual LP conversion

Primal LP :	**Dual LP** :
$\max c_1x_1 + \cdots + c_nx_n$	$\min b_1y_1 + \cdots + b_my_m$
$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$	$a_{11}y_1 + \cdots + a_{m1}y_m \leq c_1$
$\vdots$	$\vdots$
$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$	$a_{n1}y_1 + \cdots + a_{nm}y_m \leq c_n$
$x_1, \cdots, x_n \geq 0$	$y_1, \cdots, y_m \geq 0$

Example of Duality

- ☐ Why do we consider duality?
  - Sometimes the dual problem is easier to solve than the primal problem.
  - To gain new insights
  - Note: duality does not make one solve the problem more efficiently.
- ☐ **Maximum** flow problem vs. **Minimum** cut problem
- ☐ **Shortest** path problem vs. **Longest** distance problem