# CS311 Data Structures
# Lecture 08 — AVL tree

Jyh-Ming Lien

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# Logistics

## Reading

- Weiss Ch 19.1-3 BSTs
- Weiss Ch 19.4: AVL Trees

## Today

- Tree Rotations: Balancing via pointer manipulation
- AVL Trees

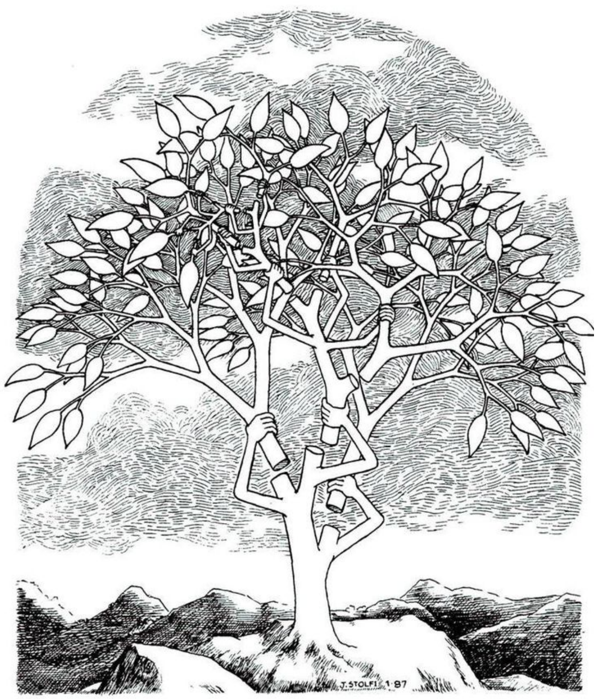
# Why Worry About Insertion and Removal?

- Q: Why worry about `insert/remove` messing with the tree? What affect can it have on the performance of future ops on the tree?
- Q: What property of a tree dictates the runtime complexity of its operations?
- Recall from our practice footnotesize
  - Build and draw a BST by inserting these numbers in order: 5, 13, 18, 21, 23, 31, 5,7 89, 130
  - Build and draw a BST by inserting these numbers in order: 31, 18, 130, 89, 21, 5, 57, 13, 23

# Balancing Trees

- `add/remove/find` complexity *O(height(t))*
- Degenerate tree has height *N*: a linked list
- Prevent this by re-balancing on `insert/remove`
- Several kinds of trees do this

| | |
|---|---|
| AVL | left/right subtree height differ by max 1 |
| Red-black | preserve 4 red/black node properties |
| AA | red-black tree + all left nodes black |
| Splay | amoritized bound on ops, very different |



# The AVL Tree

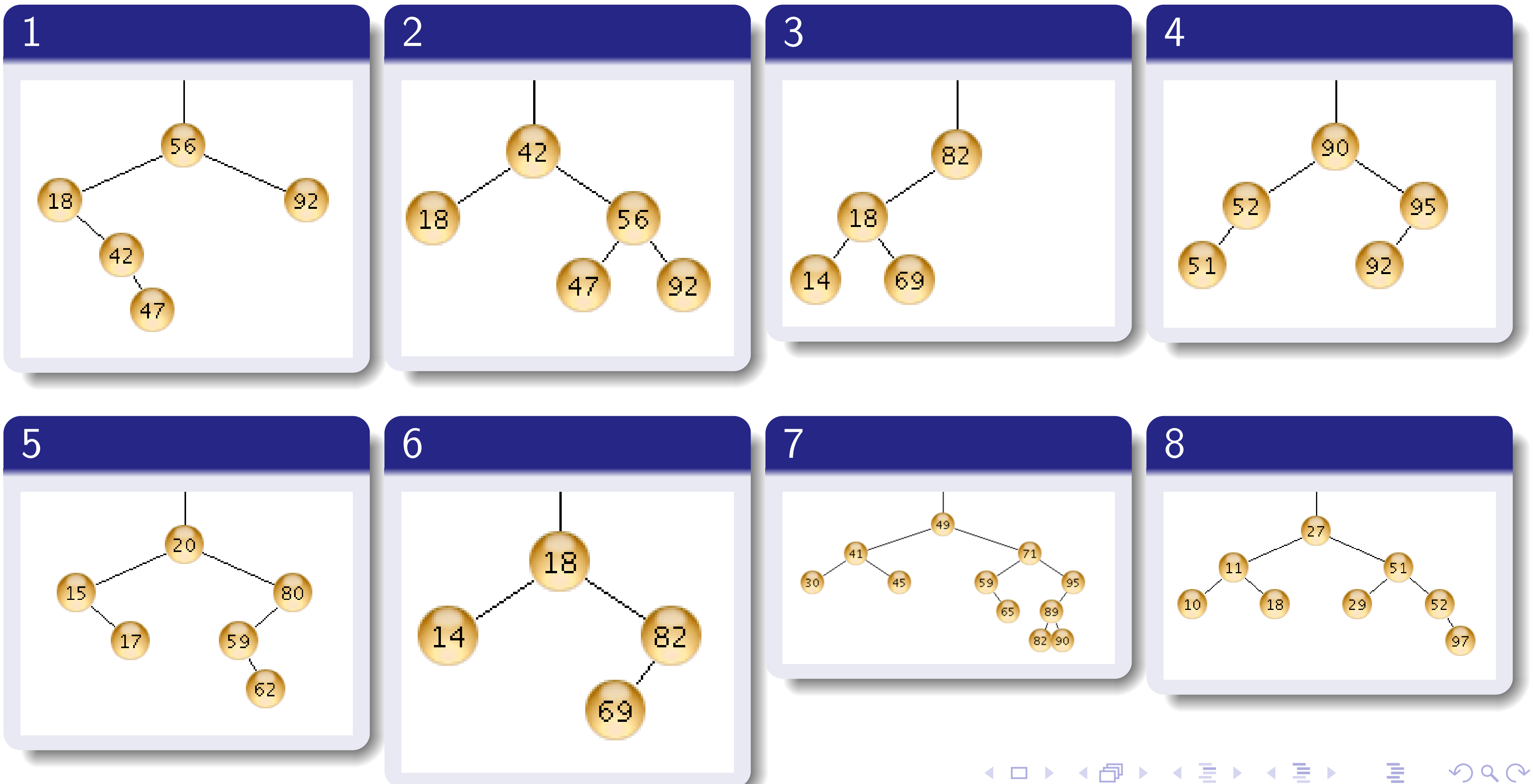
*The AVL tree is named after its two Soviet inventors, Georgy Adelson-Velsky and E. M. Landis, who published it in their 1962 paper "An algorithm for the organization of information". – Wikip: AVL Tree*

- A self-balancing tree
- Operations
- Proof of logarithmic height

# AVL Balance Property

`T` is an AVL tree if and only if

- `T.left` and `T.right` differ in height by at most 1
- AND `T.left` and `T.right` are AVL trees

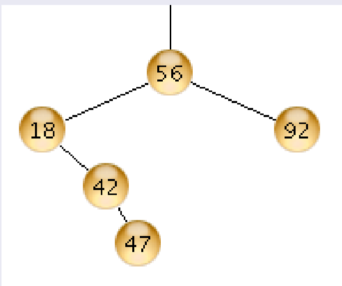


# Answers

T is an AVL tree if and only if

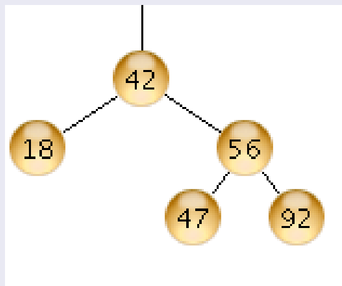
- `T.left` and `T.right` differ in height by at most 1
- AND `T.left` and `T.right` are AVL trees

### 1 Not AVL

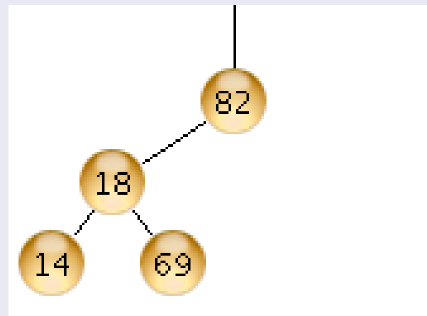



Left 3, Right 1

### 2 AVL

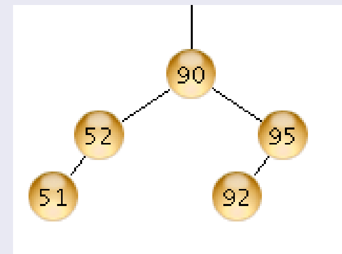



### 3 Not AVL

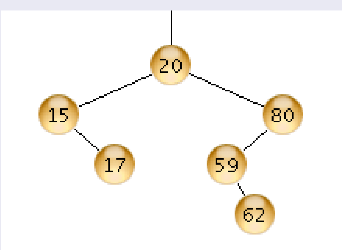



Left 2, Right 0

### 4 AVL

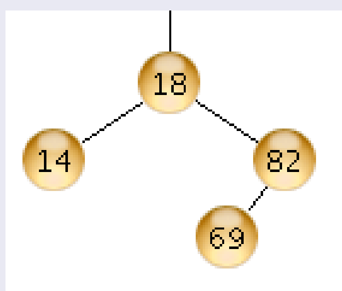



### 5 Not AVL

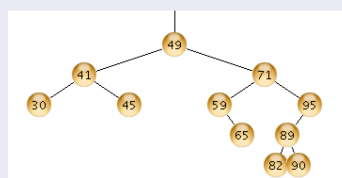



80 not AVL

### 6 AVL

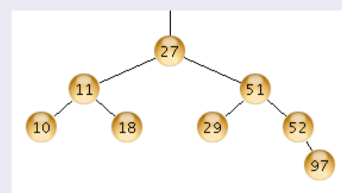



### 7 Not AVL




Left 2, Right 4
95 not AVL

### 8 AVL



# Nodes and Balancing in AVL Trees

Track *Balance Factor* of trees

- `balance = height(t.left) - height(t.right);`
- Must be -1, 0, or +1 for AVL
- If -2 or +2, must fix

```
class Node<T>{
  Node<t> left,right;
  T data;
  int height;
}
```

Don't explicitly calculate `height`

- Adjust balance factor on `insert/delete`
- Recurse down to add/remove node
- Unwind recursion up to adjust balance of ancestors
- When unbalanced, rotate to adjust heights
- Single or Double rotation can *always* adjust heights by 1

# Rotations

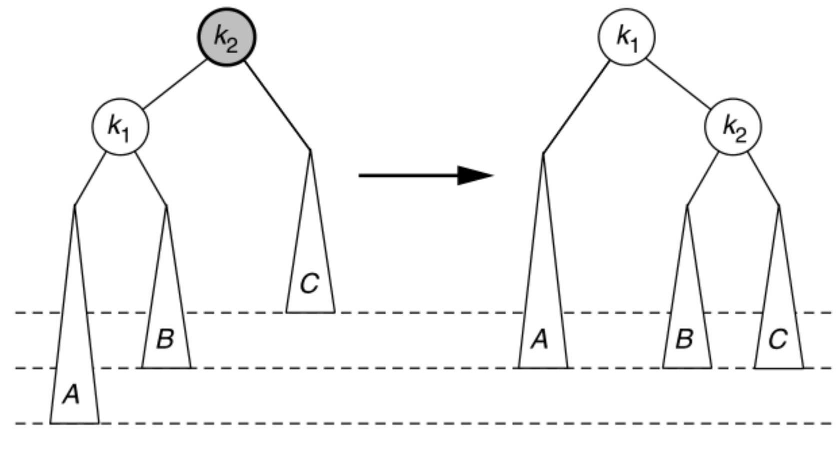
Re-balancing usually involves

- Drill down during `insert/remove`
- Follow path back up to make adjustments
- Adjustments even out height of subtrees
- Adjustments are usually rotations
- Rotation changes structure of tree without affecting ordering

# Single Rotation Basics

## Right Rotation

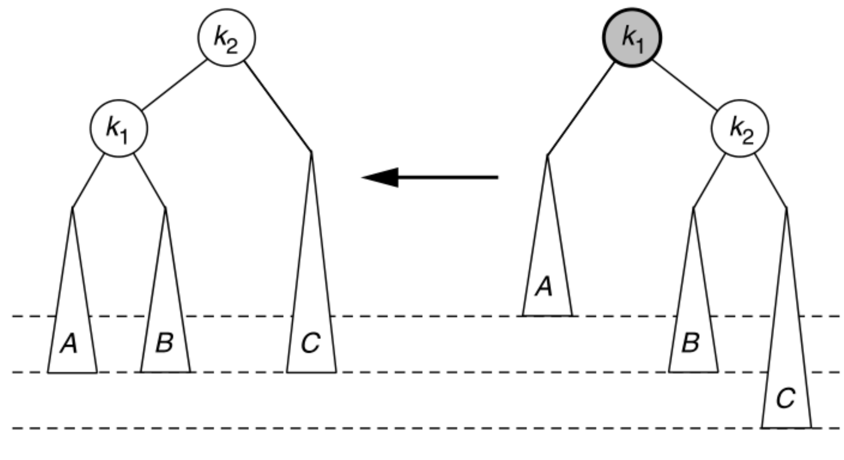
Rotation node becomes the right subtree




(a) Before rotation (b) After rotation

## Left Rotation

Rotation node becomes the left subtree

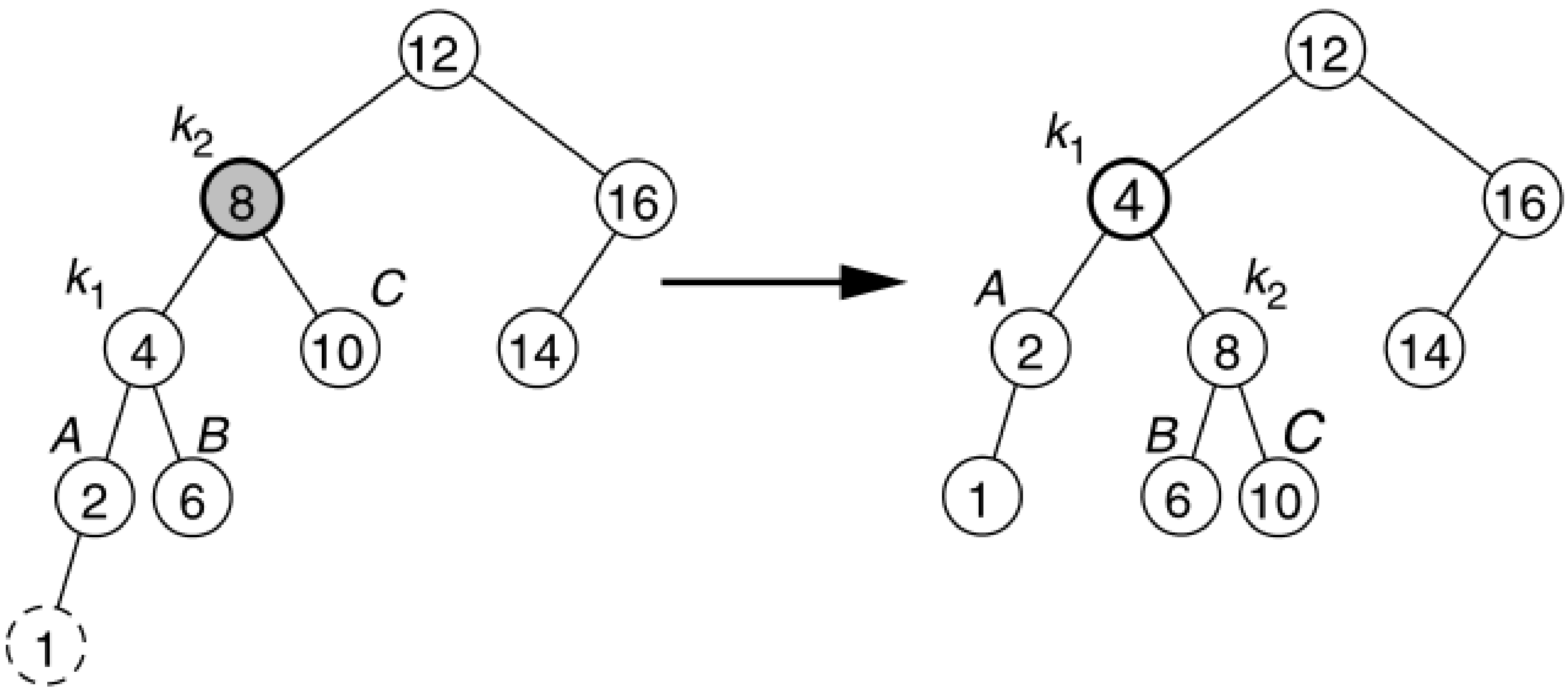



(a) After rotation (b) Before rotation

# Fixing an Insertion with a Single Rotation

Insert 1, perform rotation to balance heights

- Right rotation at 8

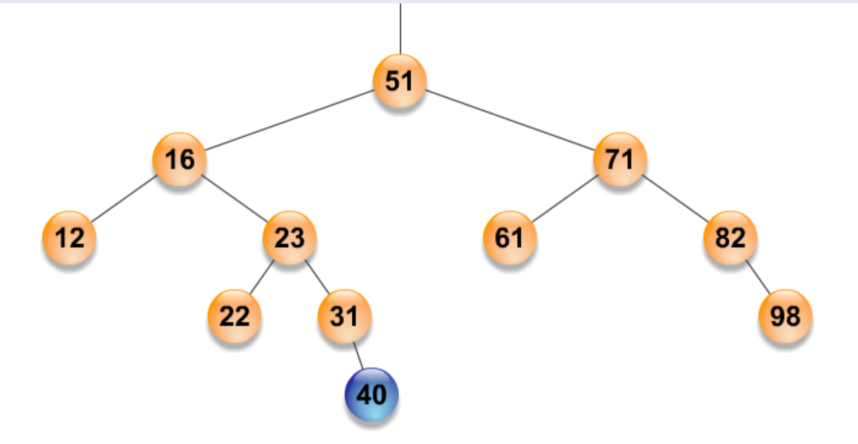



(a) Before rotation (b) After rotation

# Single Rotation Practice

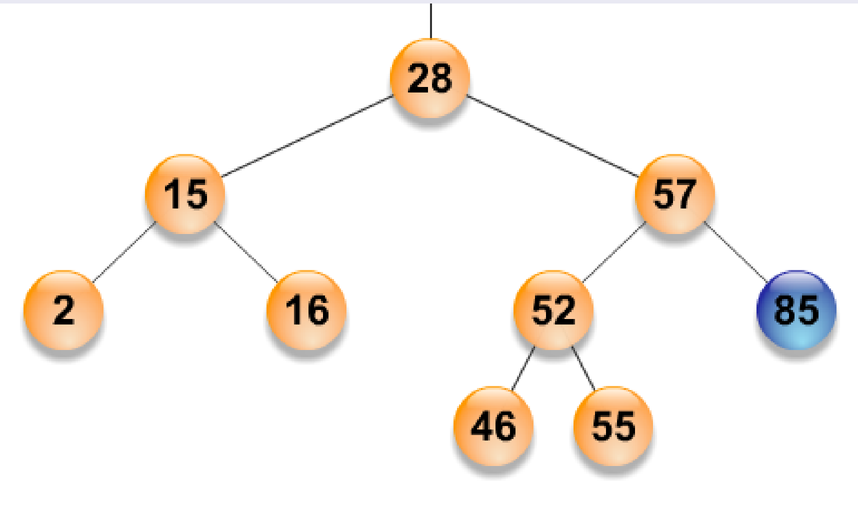
## Problem 1

- 40 was just inserted
- Rebalance tree rooted at 16
- Left-rotate 16



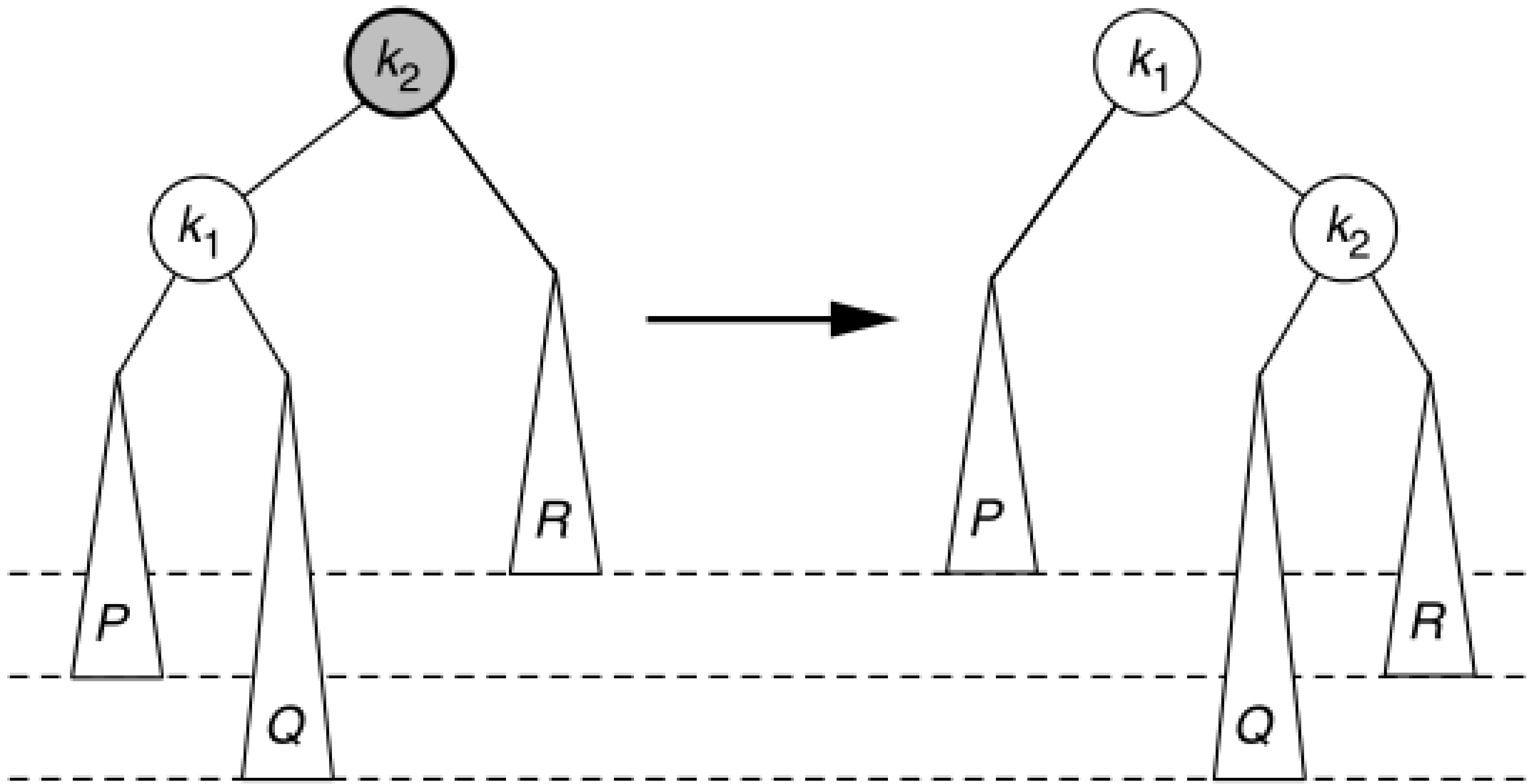

## Problem 2

- 85 is being removed
- Rebalance tree rooted at 57
- Right rotate 57



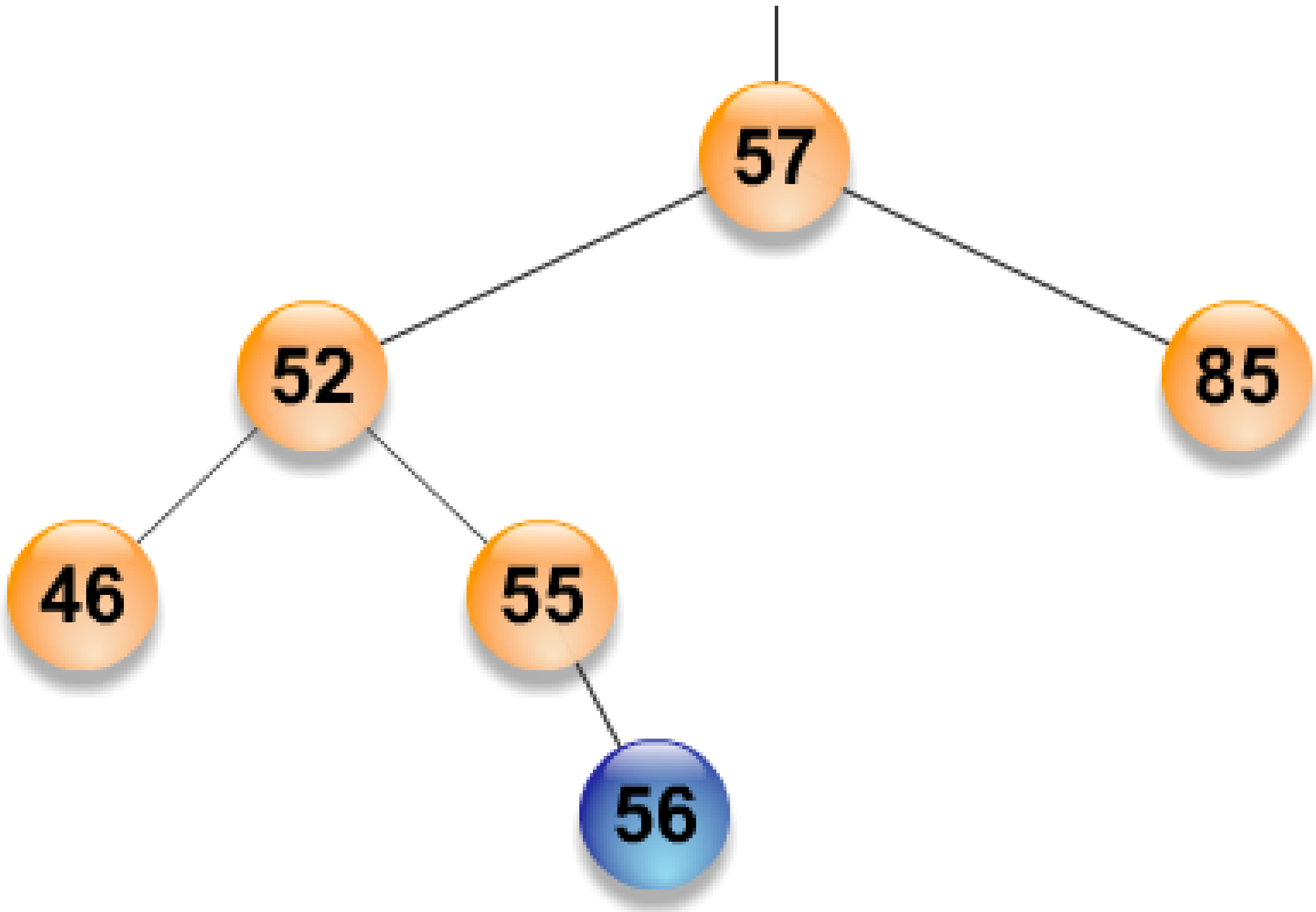
# Single Rotations Aren't Enough

Can we fix the following with a single rotation?




(a) Before rotation (b) After rotation

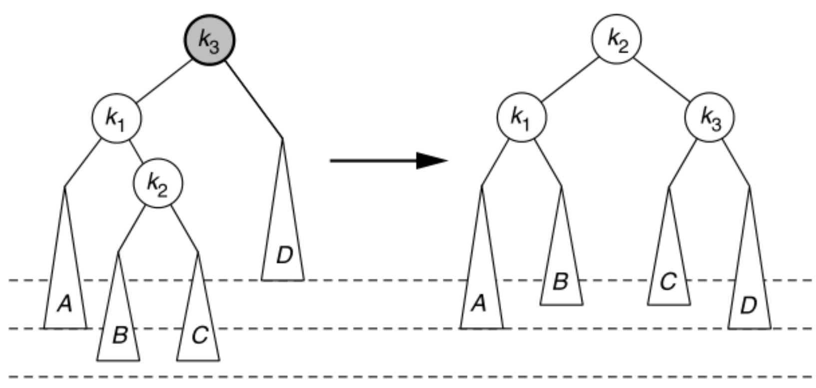
# Example: Can't fix this with single rotation



# Double Rotation Overview

## Left-Right

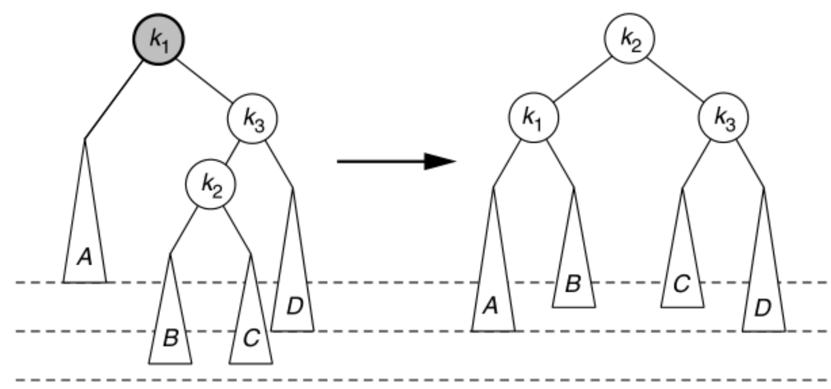
- Left Rotate at $k_1$
- Right-rotate at $k_3$




(a) Before rotation (b) After rotation

## Right-Left

- Right Rotate at $k_3$
- Left Rotate at $k_1$

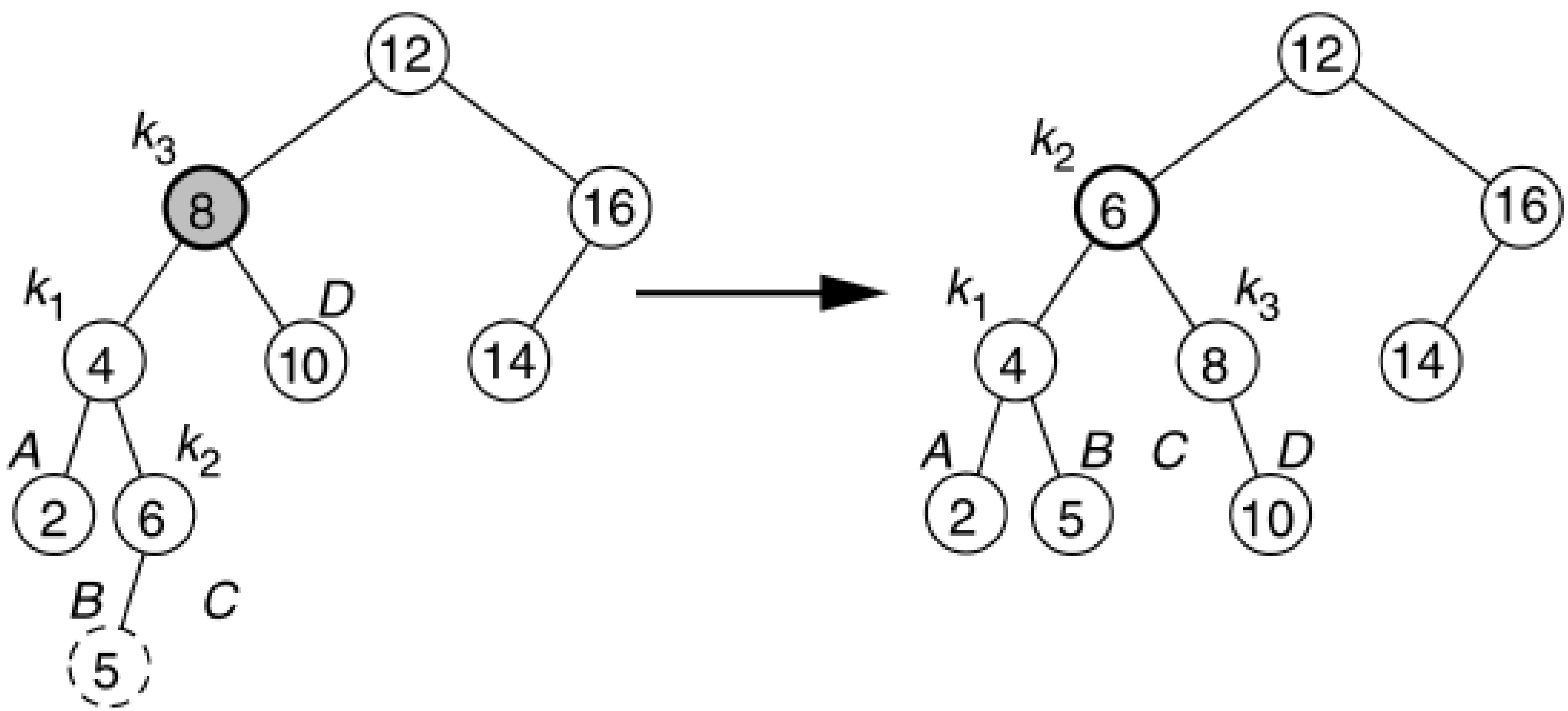



(a) Before rotation (b) After rotation

# Fixing an Insertion with a Double Rotation

Insert 5, perform two rotations to balance heights

- Problem is at 8: left height 3, right height 1
- Left rotate 4 (height imbalance remains)
- Right rotate 8 (height imbalance fixed)

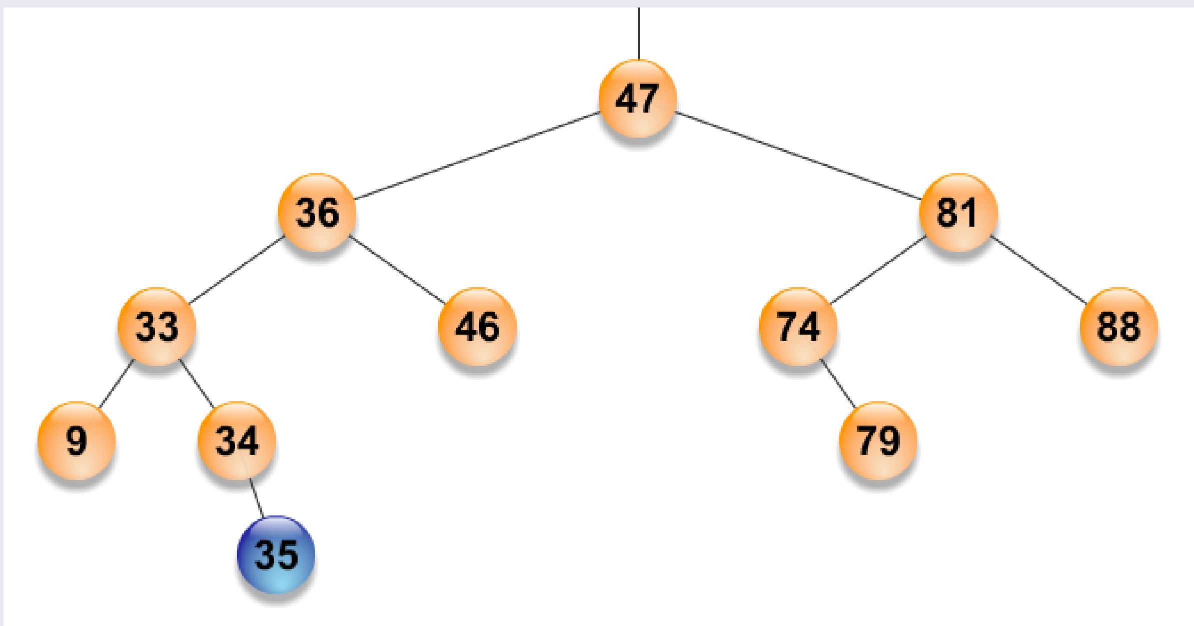



(a) Before rotation (b) After rotation

# Double Rotation Practice

## Problem 3

- 35 was just inserted
- Rebalance the tree rooted at 36
- Use two rotations, at 33 and 36
- 36 should move



# Code for Rotations?

```
class Node<T>{
  Node<T> left, right;
  T data;
  int height;
}
```

Write the following codes for single/double rotations:

```
// Single Right rotation
// t becomes right child, t.left becomes new
// root which is returned
Node<T> rightRotate( Node<T> t ) { ... }

// Left-Right Double Rotation:
// left-rotate t.left, then right-rotate t
Node<T> leftRightRotate( Node<T> t ){ ... }
```

# Example Rotation Codes

```
// Single Right rotation
Node<T> rightRotate( Node<T> t ) {
  Node<T> newRoot = t.left;
  t.left = newRoot.right;
  newRoot.right = t;
  t.height = Math.max(t.left.height,
                      t.right.height)+1;
  newRoot.height = Math.max(newRoot.left.height,
                            newRoot.right.height)+1;
  return newRoot;
}

// Left-Right Double Rotation:
// left-rotate t.left, then right-rotate t
Node<T> leftRightRotate( Node<T> t ){
  t.left = leftRotate(t.left);
  return rightRotate(t);
}
```

Computational complexities of these methods?

# Rotations in During Insertion

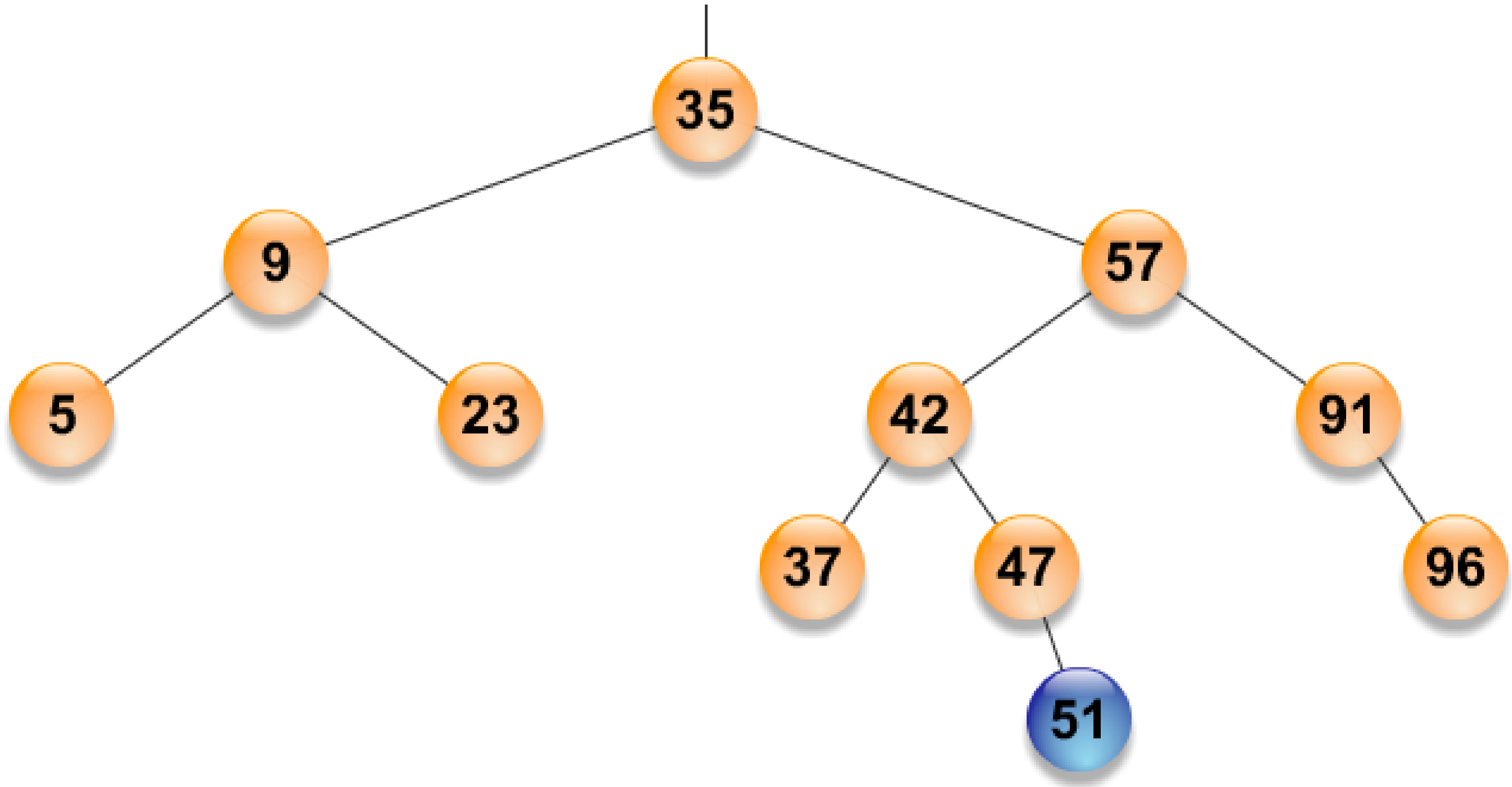
- Insertion works by first recursively inserting new data as a leaf
- Tree is "unstitched" - waiting to assign left/right branches of intermediate nodes to answers from recursive calls
- Before returning, check height differences and perform rotations if needed
- Allows left/right branches to change the nodes to which they point

# Excerpt of Insertion Code

- Identify subtree height differences to determine rotations
- Useful in removal as well

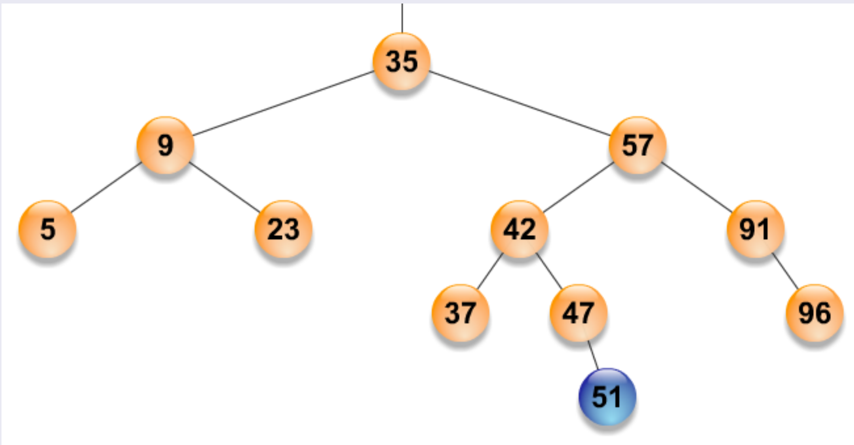
```
private AvlNode insert( Comparable x, AvlNode t ){
  if( t == null ){                                   // Found the spot to insert
    t = new AvlNode( x, null, null );                // return new node with data
  }
  else if( x.compareTo( t.element ) < 0 ) {          // Head left
    t.left = insert( x, t.left );                    //   Recursively insert
  } else{                                            // Head right
    t.right = insert( x, t.right );                  //   Recursively insert
  }                                                  //
  if(height(t.left) - height(t.right) == 2){         // t.left deeper than t.right
    if(height(t.left.left) > t.left.right) {         // outer tree unbalanced
      t = rightRotate( t );                          //   single rotation
    } else {                                         // x went left-right:
      t = leftRightRotate( t );                      //   double rotation
    }
  }
  else{ ... } // Symmetric cases for t.right deeper than t.left
  return t;
```

# Rebalance This AVL Tree

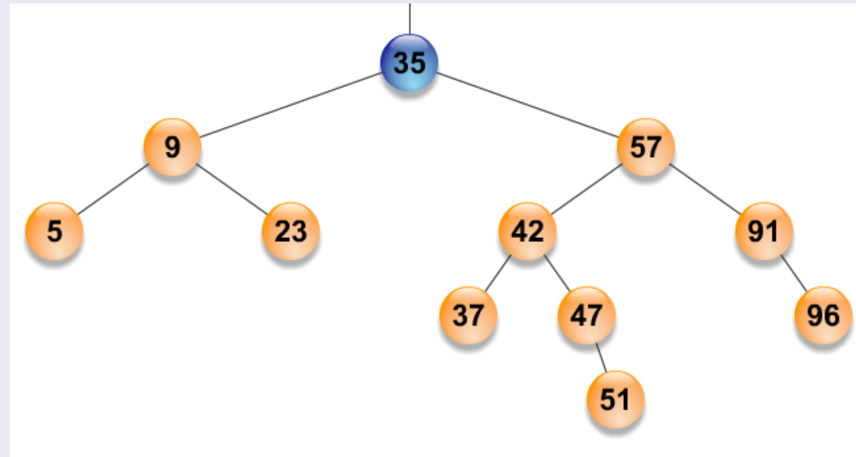



- Inserted 51
- Which node is unbalanced?
- Which rotation(s) required to fix?

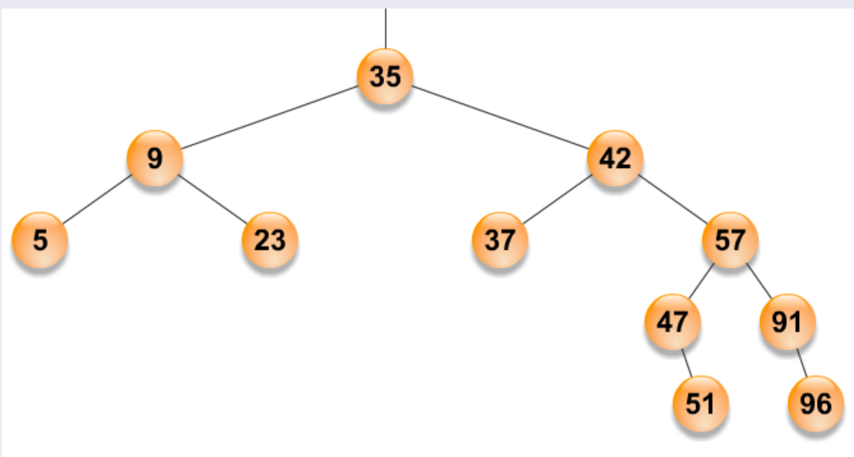
# Rebalancing Answer



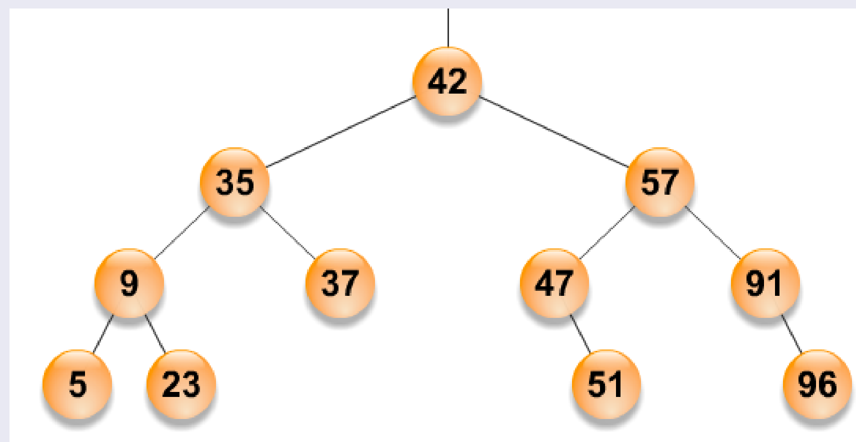

Insert 51




35 Unbalanced




After right rotate at 57




After left rotate at 35

# Does This Accomplish our Goal?

- Proposition: Maintaining the AVL Balance Property during `insert/remove` will yield a tree with $N$ nodes and height $O(\log N)$
- Prove it: What do AVL trees have to do with rabbits?

# AVL Properties Give $log(N)$ height

## Lemma (little theorem) *(Thm 19.3 in Weiss, pg 708, adapted)*

An AVL Tree of height $H$ has at least $F_{H+2} - 1$ nodes where $F_i$ is the *ith* Fibonacci number.

## Definitions

- $F_i$: *ith* Fibonacci number (0,1,1,2,3,5,8,13,...)
- $S$: size of a tree
- $H$: height (assume roots have height 1)
- $S_H$ as smallest size AVL Tree with height $H$

## Proof by Induction: Base Cases True

| Tree | height | Min Size | Calculation |
|---|---|---|---|
| empty | $H = 0$ | $S_0$ | $F_{(0+2)} - 1 = 1 - 1 = 0$ |
| root | $H = 1$ | $S_1$ | $F_{(1+2)} - 1 = 2 - 1 = 1$ |
| root+(left or right) | $H = 2$ | $S_2$ | $F_{(2+2)} - 1 = 3 - 1 = 2$ |

# Induction Part 1

## Consider an Arbitrary AVL tree $T$

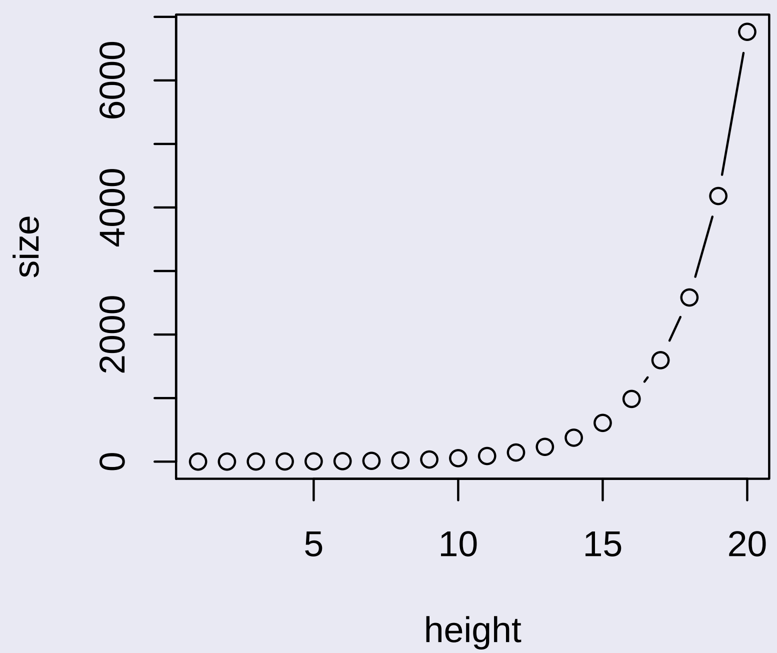
- $T$ has height $H$
- $S_H$ smallest size for tree $T$
- Show that the smallest size $S_H = F_{H+2} - 1$
- Assume equation true for smaller trees
  - Left/Right are smaller AVL trees
  - Left/Right differ in height by at most 1

# Induction Part 2

- $T$ has height $H$
- Assume for height $h < H$, smallest size of $T$ is $S_h = F_{h+2} - 1$
- Suppose Left is 1 higher than Right
- Left Height: $h = H - 1$
- Left Size: $F_{(H-1)+2} - 1 = F_{H+1} - 1$
- Right Height: $h = H - 2$
- Right Size: $F_{(H-2)+2} - 1 = F_H - 1$

$$\begin{aligned} S_H &= size(Left) + size(Right) + 1 \\ &= (F_{H+1} - 1) + (F_H - 1) + 1 \\ &= F_{H+1} + F_H - 1 \\ &= F_{H+2} - 1 \quad \blacksquare \end{aligned}$$

# Fibonacci Growth




AVL Tree of with height $H$ has at least $F_{H+2} - 1$ nodes.

- How does $F_H$ grow wrt $H$?
- Exponentially: $F_H \approx \phi^H = 1.618^H$
- $\phi$: The Golden Ratio
- So, $\log(F_H) \approx H \log(\phi)$
- Or, $\log(N) \approx height \times \phi$
- Or, $\log(size) \approx height * constant$