Logistics

Reading
- Weiss Ch 19.1-3 BSTs
- Weiss Ch 19.4: AVL Trees

Today
- Tree Rotations: Balancing via pointer manipulation
- AVL Trees
Q: Why worry about insert/remove messing with the tree? What affect can it have on the performance of future ops on the tree?

Q: What property of a tree dictates the runtime complexity of its operations?

Recall from our practice footnotesize

- Build and draw a BST by inserting these numbers in order: 5, 13, 18, 21, 23, 31, 5, 7, 89, 130
- Build and draw a BST by inserting these numbers in order: 31, 18, 130, 89, 21, 5, 57, 13, 23
Balancing Trees

- add/remove/find complexity $O(\text{height}(t))$
- Degenerate tree has height $N$: a linked list
- Prevent this by re-balancing on insert/remove
- Several kinds of trees do this
  - **AVL** left/right subtree height differ by max 1
  - **Red-black** preserve 4 red/black node properties
  - **AA** red-black tree + all left nodes black
  - **Splay** amortized bound on ops, very different
The AVL tree is named after its two Soviet inventors, Georgy Adelson-Velsky and E. M. Landis, who published it in their 1962 paper "An algorithm for the organization of information". 
– Wikip: AVL Tree

- A self-balancing tree
- Operations
- Proof of logarithmic height
AVL Balance Property

T is an AVL tree if and only if

- T.left and T.right differ in height by at most 1
- AND T.left and T.right are AVL trees
T is an AVL tree if and only if

- $T$.left and $T$.right differ in height by at most 1
- AND $T$.left and $T$.right are AVL trees
Track Balance Factor of trees
- balance = height(t.left) - height(t.right);
- Must be -1, 0, or +1 for AVL
- If -2 or +2, must fix

class Node<T>{
    Node<T> left, right;
    T data;
    int height;
}

Don’t explicitly calculate height
- Adjust balance factor on insert/delete
- Recurse down to add/remove node
- Unwind recursion up to adjust balance of ancestors
- When unbalanced, rotate to adjust heights
- Single or Double rotation can always adjust heights by 1
Rotations

Re-balancing usually involves

- Drill down during insert/remove
- Follow path back up to make adjustments
- Adjustments even out height of subtrees
- Adjustments are usually rotations
- Rotation changes structure of tree without affecting ordering
**Single Rotation Basics**

**Right Rotation**
Rotation node becomes the right subtree

![Diagram of Right Rotation](image)

(a) Before rotation  
(b) After rotation

**Left Rotation**
Rotation node becomes the left subtree

![Diagram of Left Rotation](image)

(a) After rotation  
(b) Before rotation
Fixing an Insertion with a Single Rotation

Insert 1, perform rotation to balance heights

- Right rotation at 8

(a) Before rotation

(b) After rotation
Problem 1
- 40 was just inserted
- Rebalance tree rooted at 16
- Left-rotate 16

Problem 2
- 85 is being removed
- Rebalance tree rooted at 57
- Right rotate 57
Single Rotations Aren’t Enough

Can we fix the following with a single rotation?

(a) Before rotation
(b) After rotation
Example: Can’t fix this with single rotation
Double Rotation Overview

Left-Right
- Left Rotate at $k_1$
- Right-rotate at $k_3$

Right-Left
- Right Rotate at $k_3$
- Left Rotate at $k_1$
Fixing an Insertion with a Double Rotation

Insert 5, perform two rotations to balance heights
- Problem is at 8: left height 3, right height 1
- Left rotate 4 (height imbalance remains)
- Right rotate 8 (height imbalance fixed)
Problem 3

- 35 was just inserted
- Rebalance the tree rooted at 36
- Use two rotations, at 33 and 36
- 36 should move
class Node<T>{
    Node<T> left, right;
    T data;
    int height;
}

Write the following codes for single/double rotations:

// Single Right rotation
// t becomes right child, t.left becomes new
// root which is returned
Node<T> rightRotate( Node<T> t ) { ... }

// Left-Right Double Rotation:
// left-rotate t.left, then right-rotate t
Node<T> leftRightRotate( Node<T> t ){ ... }
Example Rotation Codes

// Single Right rotation
Node<T> rightRotate( Node<T> t ) {
    Node<T> newRoot = t.left;
    t.left = newRoot.right;
    newRoot.right = t;
    t.height = Math.max(t.left.height,
                        t.right.height)+1;
    newRoot.height = Math.max(newRoot.left.height,
                         newRoot.right.height)+1;
    return newRoot;
}

// Left-Right Double Rotation:
// left-rotate t.left, then right-rotate t
Node<T> leftRightRotate( Node<T> t ){
    t.left = leftRotate(t.left);
    return rightRotate(t);
}

Computational complexities of these methods?
Rotations in During Insertion

- Insertion works by first recursively inserting new data as a leaf
- Tree is "unstitched" - waiting to assign left/right branches of intermediate nodes to answers from recursive calls
- Before returning, check height differences and perform rotations if needed
- Allows left/right branches to change the nodes to which they point
• Identify subtree height differences to determine rotations
• Useful in removal as well

private AvlNode insert( Comparable x, AvlNode t ){
    if( t == null ){ // Found the spot to insert
        t = new AvlNode( x, null, null ); // return new node with data
    } 
    else if( x.compareTo( t.element ) < 0 ) { // Head left
        t.left = insert( x, t.left ); // Recursively insert
    } else{ // Head right
        t.right = insert( x, t.right ); // Recursively insert
    } 
    if(height(t.left) - height(t.right) == 2){ // t.left deeper than t.right
        if(height(t.left.left) > t.left.right) { // outer tree unbalanced
            t = rightRotate( t ); // single rotation
        } else { // x went left-right:
            t = leftRightRotate( t ); // double rotation
        } 
    } 
    else{ ... } // Symmetric cases for t.right deeper than t.left
    return t;
}
Inserted 51
Which node is unbalanced?
Which rotation(s) required to fix?
Rebalancing Answer

Insert 51

35 Unbalanced

After right rotate at 57

After left rotate at 35
Proposition: Maintaining the AVL Balance Property during insert/remove will yield a tree with $N$ nodes and height $O(\log N)$

Prove it: What do AVL trees have to do with rabbits?
AVL Properties Give $\log(N)$ height

Lemma (little theorem) (*Thm 19.3 in Weiss, pg 708, adapted*)

An AVL Tree of height $H$ has at least $F_{H+2} - 1$ nodes where $F_i$ is the $ith$ Fibonacci number.

Definitions

- $F_i$: $ith$ Fibonacci number (0, 1, 1, 2, 3, 5, 8, 13, ...)
- $S$: size of a tree
- $H$: height (assume roots have height 1)
- $S_H$ as smallest size AVL Tree with height $H$

Proof by Induction: Base Cases True

<table>
<thead>
<tr>
<th>Tree</th>
<th>height</th>
<th>Min Size</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>empty</td>
<td>$H = 0$</td>
<td>$S_0$</td>
<td>$F_{(0+2)} - 1 = 1 - 1 = 0$</td>
</tr>
<tr>
<td>root</td>
<td>$H = 1$</td>
<td>$S_1$</td>
<td>$F_{(1+2)} - 1 = 2 - 1 = 1$</td>
</tr>
<tr>
<td>root+(left or right)</td>
<td>$H = 2$</td>
<td>$S_2$</td>
<td>$F_{(2+2)} - 1 = 3 - 1 = 2$</td>
</tr>
</tbody>
</table>
Consider an Arbitrary AVL tree $T$

- $T$ has height $H$
- $S_H$ smallest size for tree $T$
- Show that the smallest size $S_H = F_{H+2} - 1$
- Assume equation true for smaller trees
  - Left/Right are smaller AVL trees
  - Left/Right differ in height by at most 1
Induction Part 2

- $T$ has height $H$
- Assume for height $h < H$, smallest size of $T$ is
  
  \[ S_h = F_{h+2} - 1 \]
- Suppose Left is 1 higher than Right
- Left Height: $h = H - 1$
- Left Size:
  \[ F_{(H-1)+2} - 1 = F_{H+1} - 1 \]
- Right Height: $h = H - 2$
- Right Size:
  \[ F_{(H-2)+2} - 1 = F_H - 1 \]

\[
S_H = \text{size}(\text{Left}) + \text{size}(\text{Right}) + 1
= (F_{H+1} - 1) + (F_H - 1) + 1
= F_{H+1} + F_H - 1
= F_{H+2} - 1 \quad \blacksquare
\]
AVL Tree of with height $H$ has at least $F_{H+2} - 1$ nodes.

- How does $F_H$ grow wrt $H$?
- Exponentially:
  \[ F_H \approx \phi^H = 1.618^H \]
- $\phi$: The Golden Ratio
- So, $\log(F_H) \approx H \log(\phi)$
- Or, $\log(N) \approx \text{height} \times \phi$
- Or,
  \[ \log(\text{size}) \approx \text{height} \times \text{constant} \]