Barycentric Coordinates

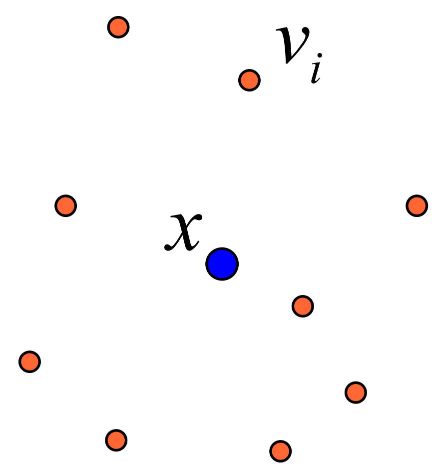
Adopted from Ju Tao and Scott Schaefer's lecture notes

Coordinates

- Homogeneous coordinates
  - Given points $v_\Sigma = \{v_1, \cdots, v_i, \cdots\}$
  - Express a new point x as affine combination of v_Σ

$$\boxed{x = \sum b_i v_i, \quad where \sum b_i = 1}$$

  - b_i are called *homogeneous coordinates*
  - *Barycentric* if all $b_i \geq 0$



Applications

- Boundary interpolation

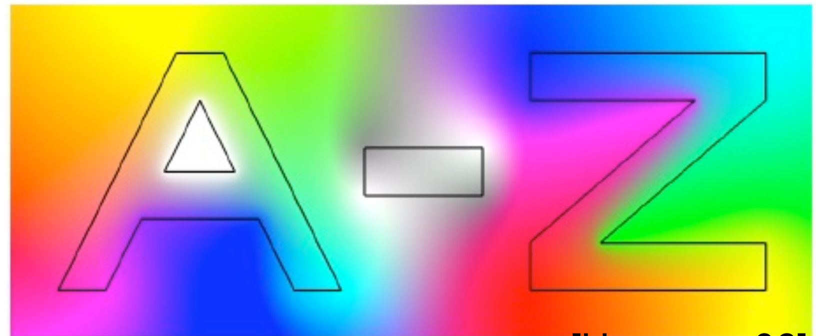
$$f(x) = \sum b_i f_i$$

  – Color/Texture interpolation

- Mapping

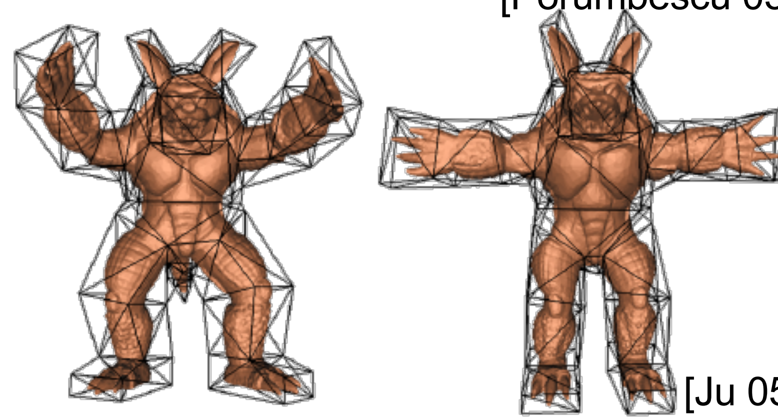
$$x' = \sum b_i v'_i$$

  – Shell texture
  – Image/Shape deformation



[Hormann 06]

[Porumbescu 05]



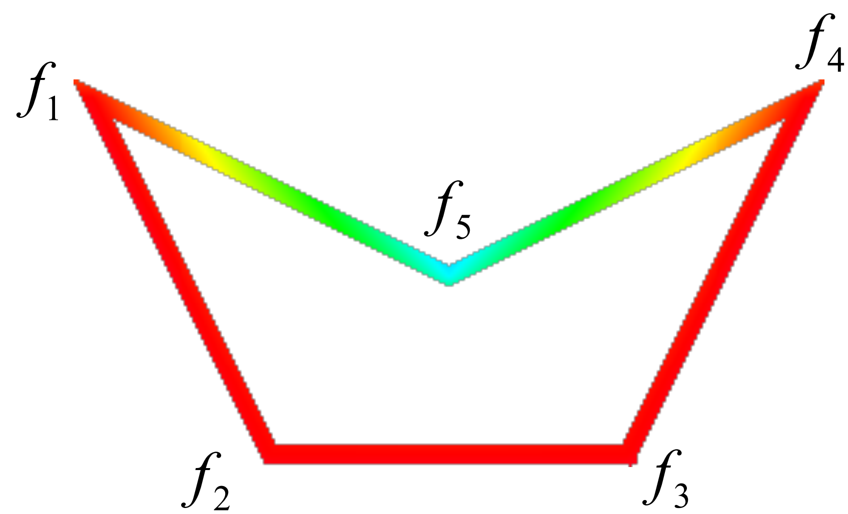
[Ju 05]

Boundary Value Interpolation

- Given p_i, compute w_i such that $v = \frac{\sum_i w_i p_i}{\sum_i w_i}$
- Given values f_i at p_i, construct a function

$$\hat{f}(v) = \frac{\sum_i w_i f_i}{\sum_i w_i}$$

- Interpolates values at vertices
- Linear on boundary
- Smooth on interior



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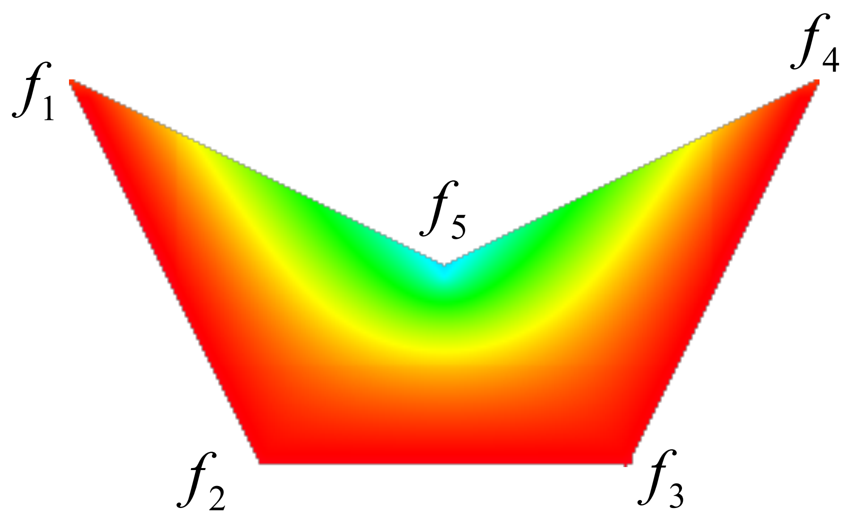
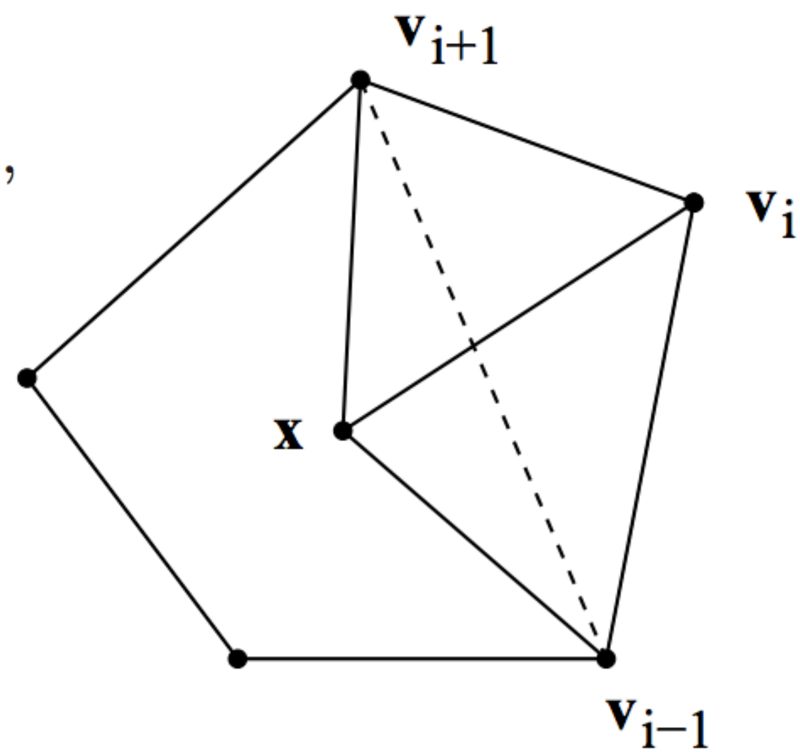


Wachspress Coordinates

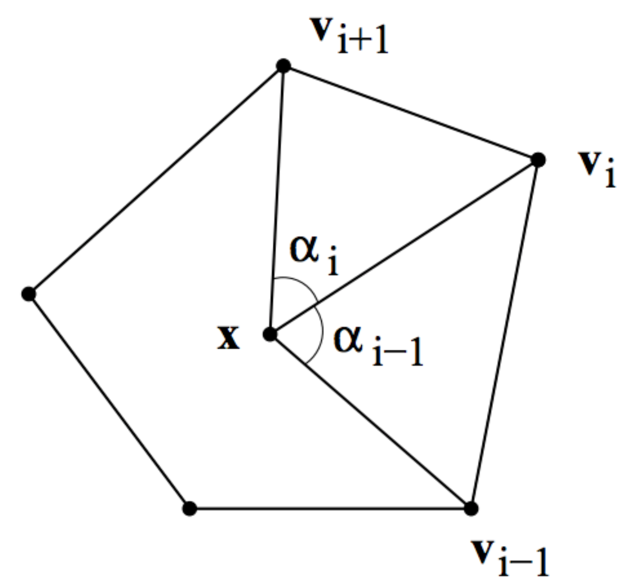
$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})},$$

$$w_i(\mathbf{x}) = \frac{A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1})}{A(\mathbf{x}, \mathbf{v}_{i-1}, \mathbf{v}_i) A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1})},$$



Mean Value Coordinates

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{|\mathbf{v}_i - \mathbf{x}|},$$



General Construction

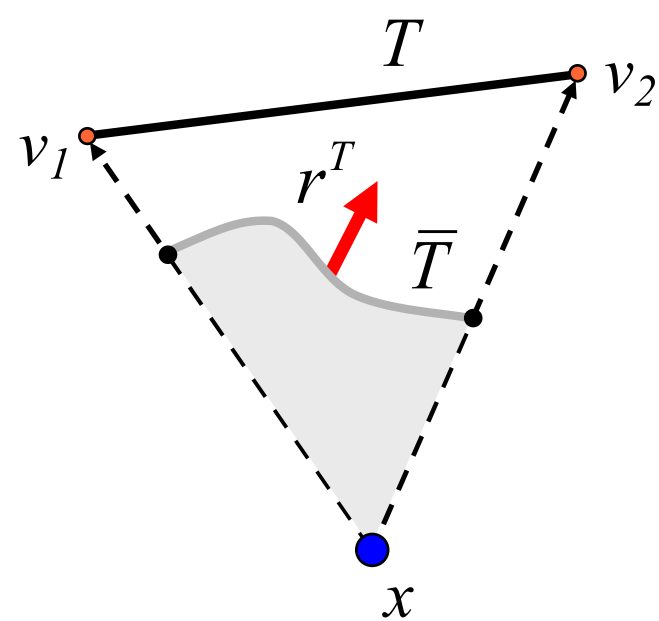
- Instead of a circle, pick any closed curve G

1. Project each edge $T=\{v_1,v_2\}$ of the polygon onto a curve segment $\bar{T}$ on G.
2. Write the **integral of outward unit normal** of each arc, r^T, using the two vectors:

$$r^T = u_1^T(v_1 - x) + u_2^T(v_2 - x)$$

3. The integral of outward unit normal over any closed curve is zero (Stoke's Theorem). So the following weights are homogeneous:

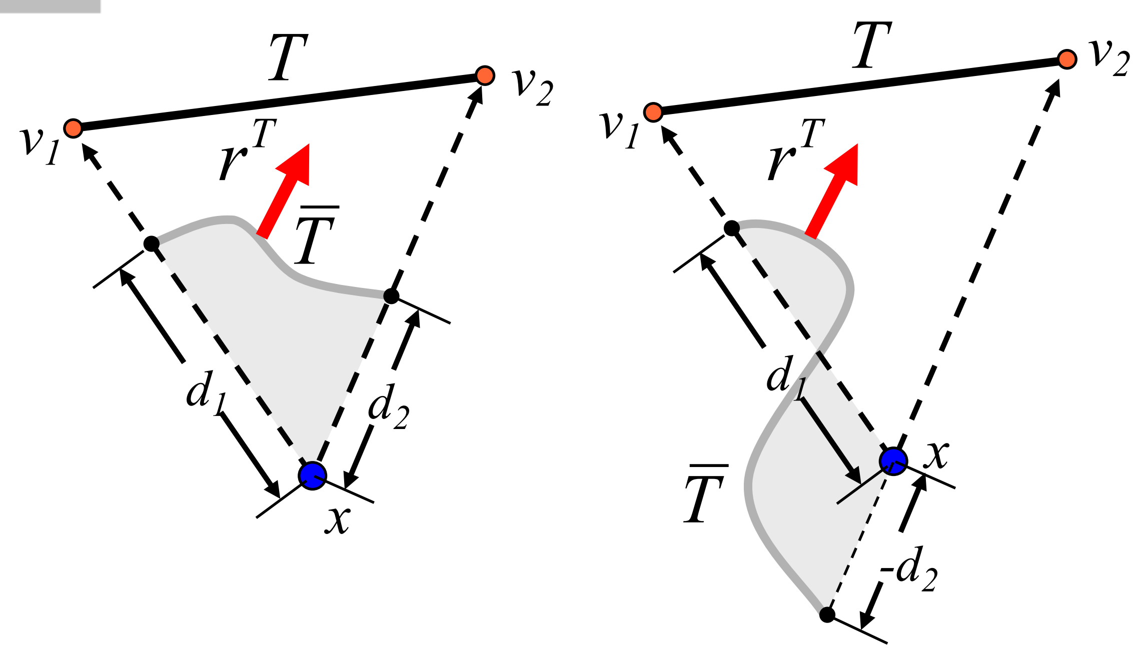
$$w_i = \sum_{T: v_i \in T} u_i^T$$



Our General Construction

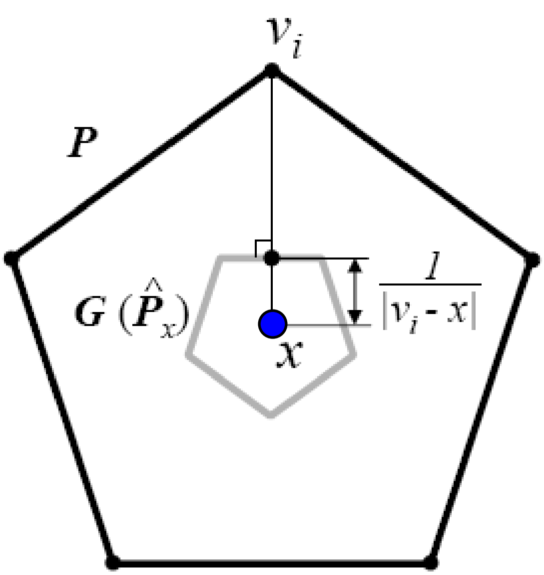
- To obtain r^T:
  - Apply Stoke's Theorem

$$\boxed{r^T = d_1 n_1^T + d_2 n_2^T}$$

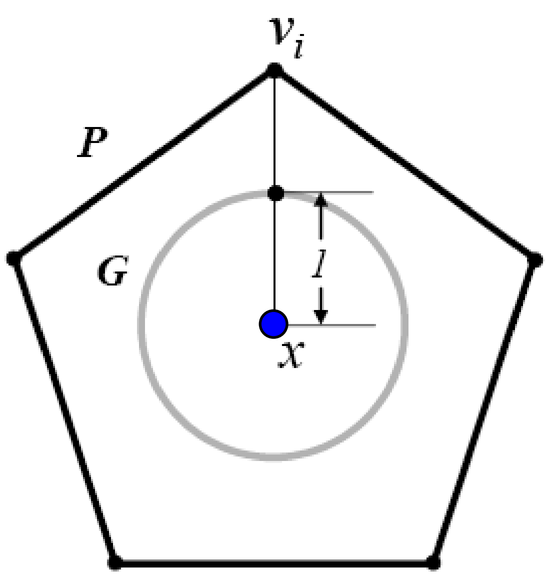


Examples

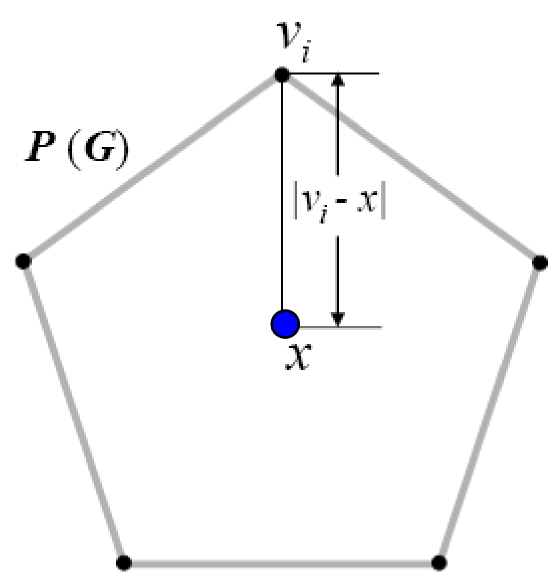
- Some interesting G result in known coordinates
  - We call G the *generating curve*



Wachspress
(G is the polar dual)



Mean value
(G is the unit circle)



Discrete harmonic
(G is the original polygon)

General Construction in 3D

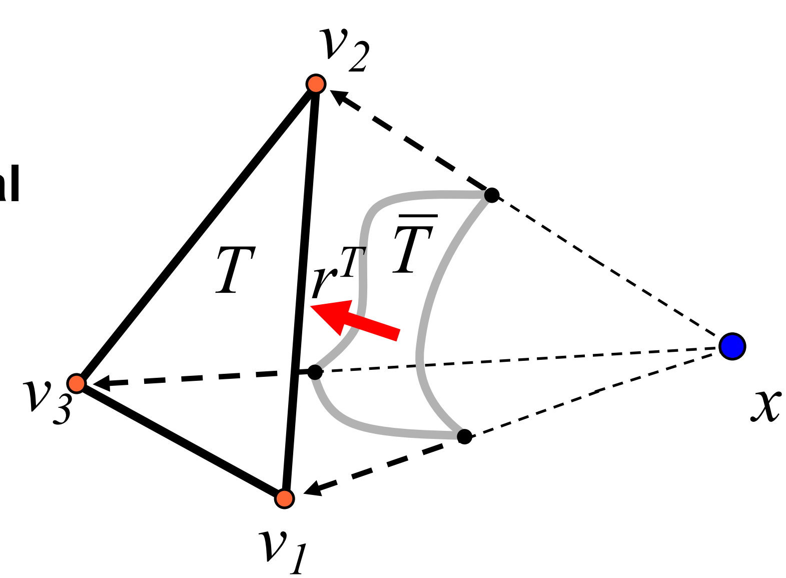
- Pick any closed generating surface G

1. Project each triangle $T = \{v_1, v_2, v_3\}$ of the polyhedron onto a surface patch $\bar{T}$ on G.
2. Write the **integral of outward unit normal** of each patch, r^T, using three vectors:

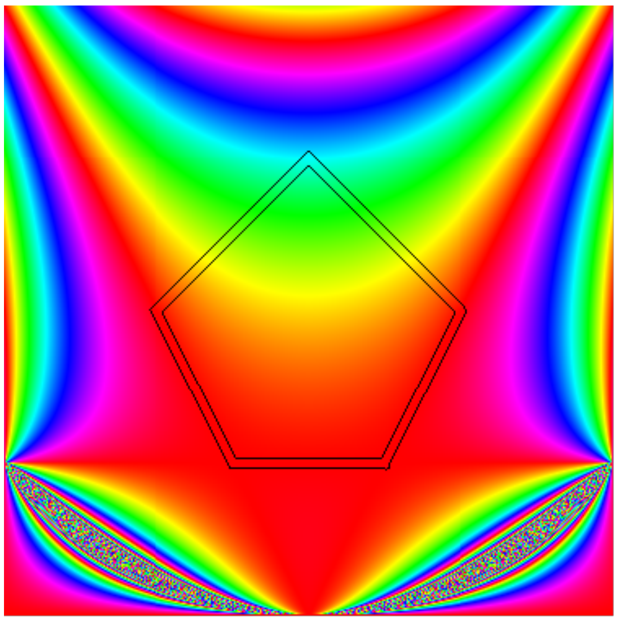
$$\boxed{r^T = \sum_{i=1}^{3} u_i^T (v_i - x)}$$

3. The integral of outward unit normal over any closed surface is zero. So the following weights are homogeneous:

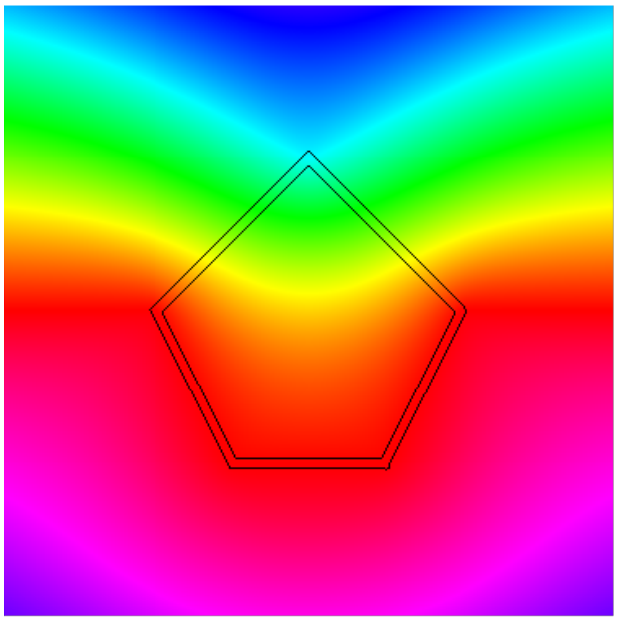
$$\boxed{w_i = \sum_{T: v_i \in T} u_i^T}$$



Comparison



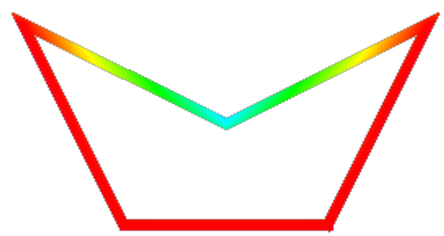
convex polygons
(Wachspress Coordinates)



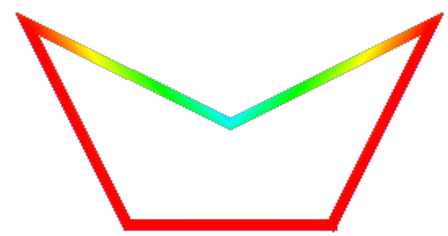
closed polygons
(Mean Value Coordinates)

Comparison-Non-convex polygons

- Boundary interpolation

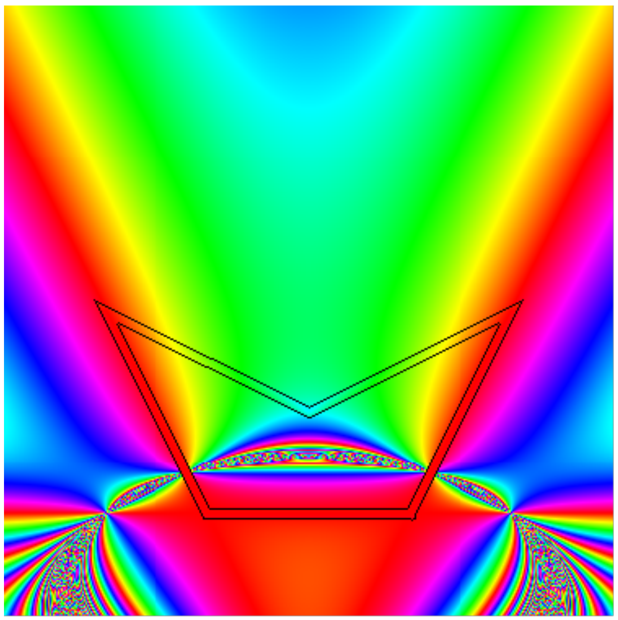


Non-convex polygons
(Wachspress Coordinates)

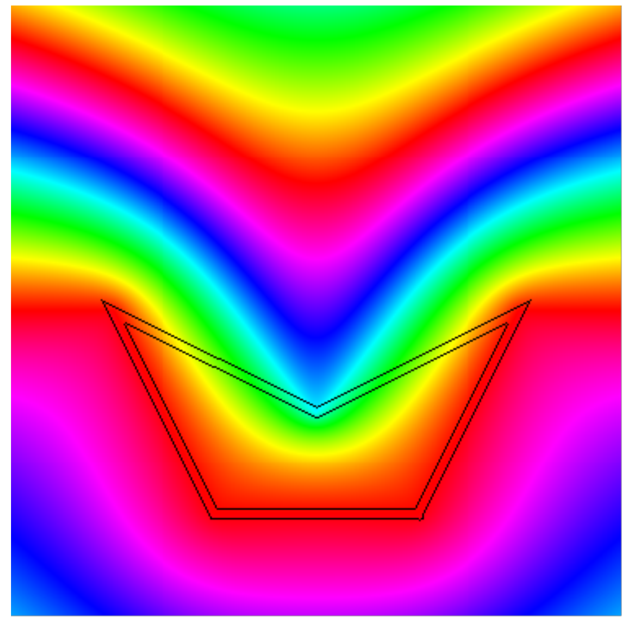


Non-convex polygons
(Mean Value Coordinates)

Comparison-Non-convex polygons

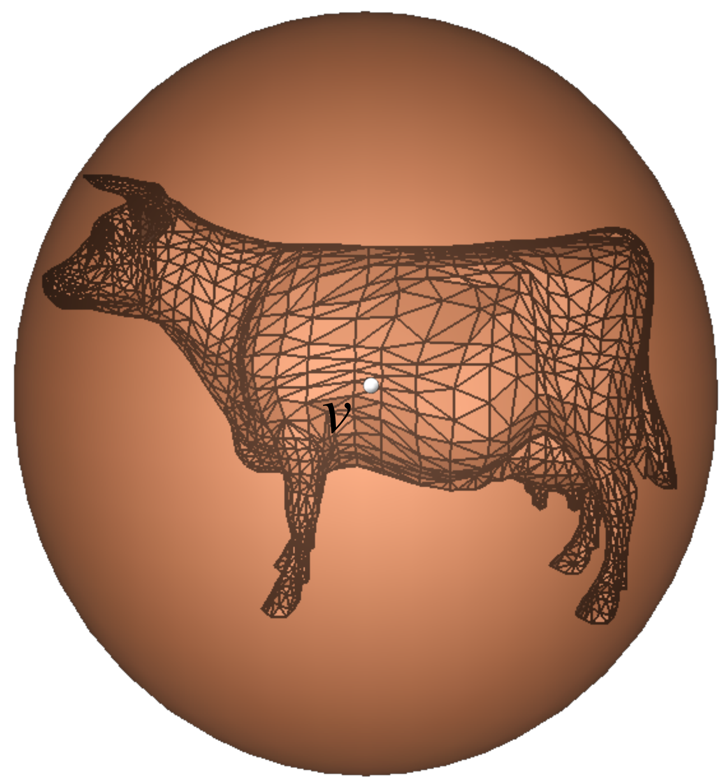


Non-convex polygons
(Wachspress Coordinates)



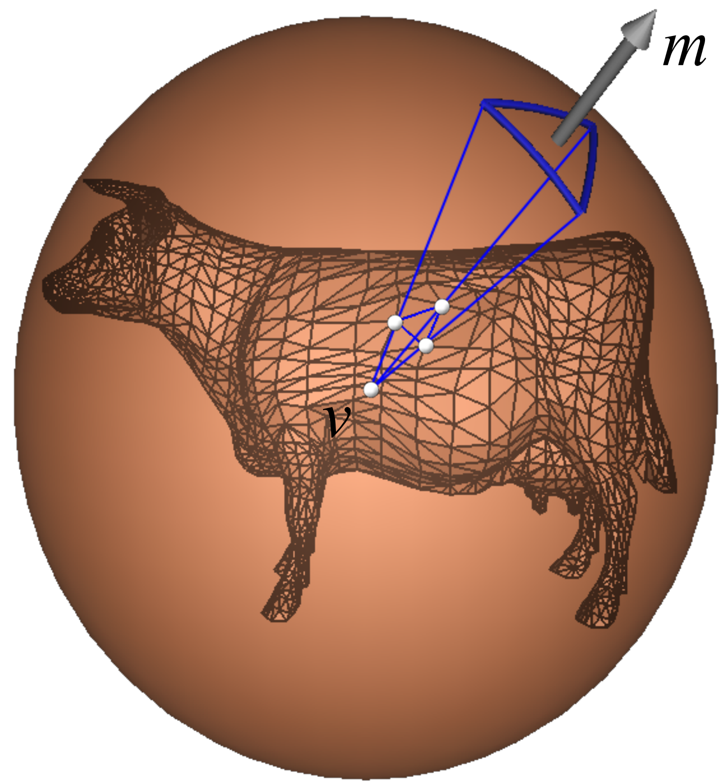
Non-convex polygons
(Mean Value Coordinates)

3D Mean Value Coordinates



3D Mean Value Coordinates

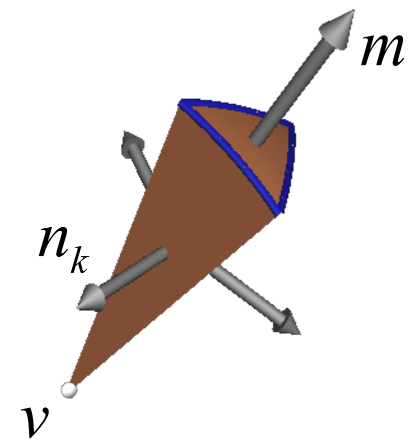
- Exactly same as 2D but must compute mean vector m for a given spherical triangle



3D Mean Value Coordinates

- Exactly same as 2D but must compute mean vector m for a given spherical triangle
- Build wedge with face normals n_k

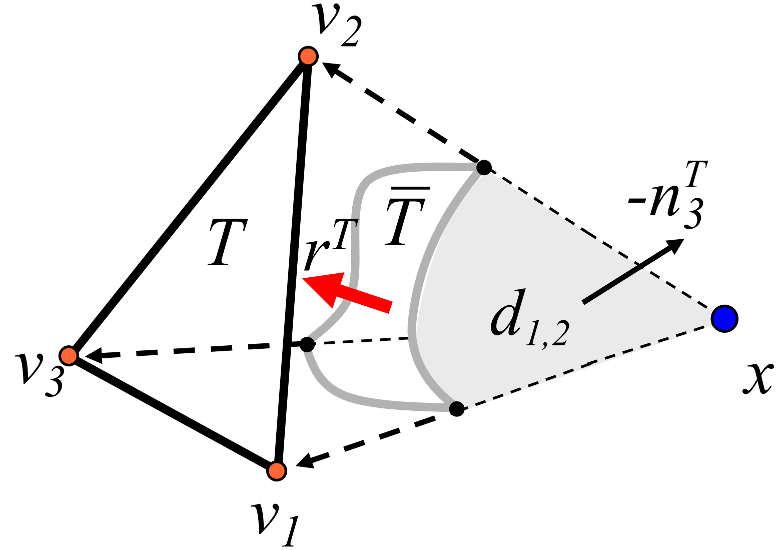
$$\sum_{k=1}^{3} \tfrac{1}{2}\theta_k n_k + m = 0$$



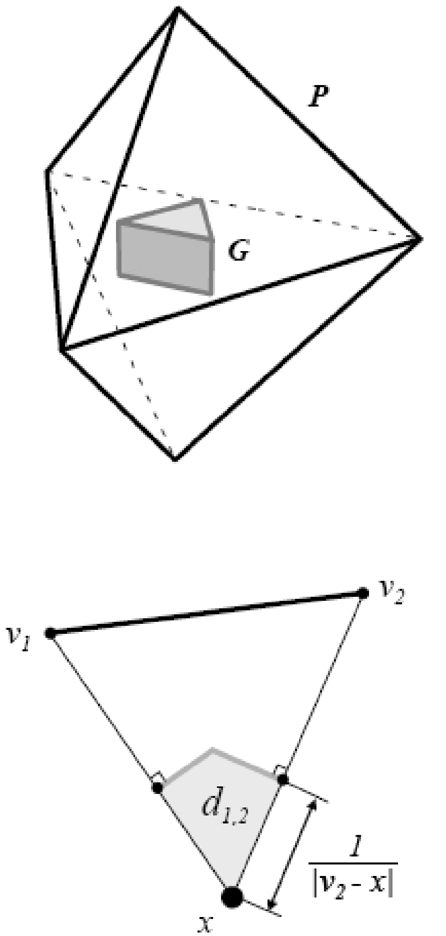
General Construction in 3D

- To obtain r^T:
  - Apply Stoke's Theorem

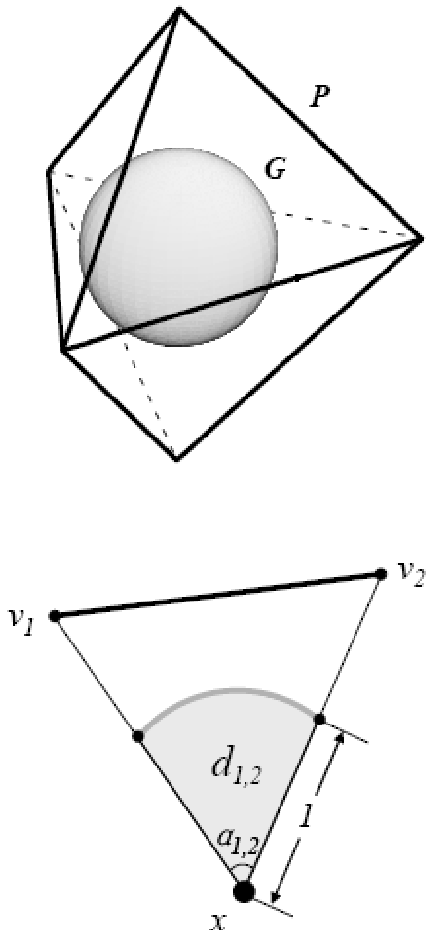
$$r^T = \sum_{i=1}^{3} d_{i-1,i+1} n_i^T$$



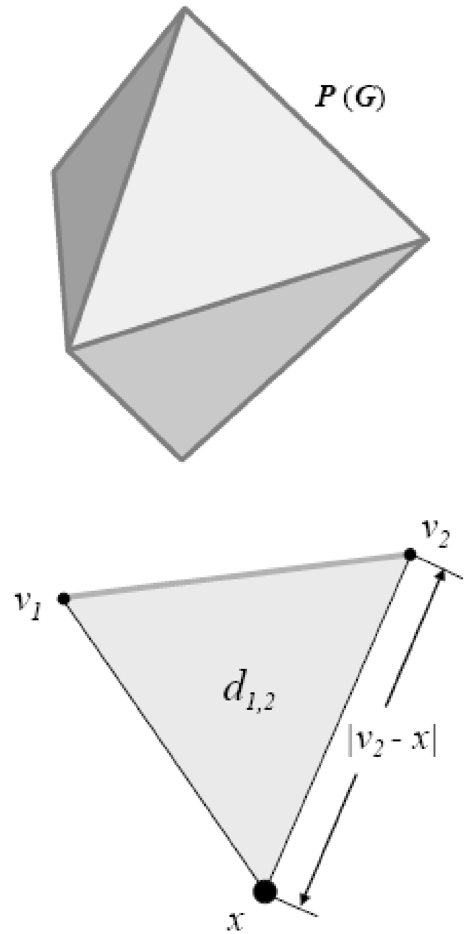
Examples



Wachspress
(G: polar dual)



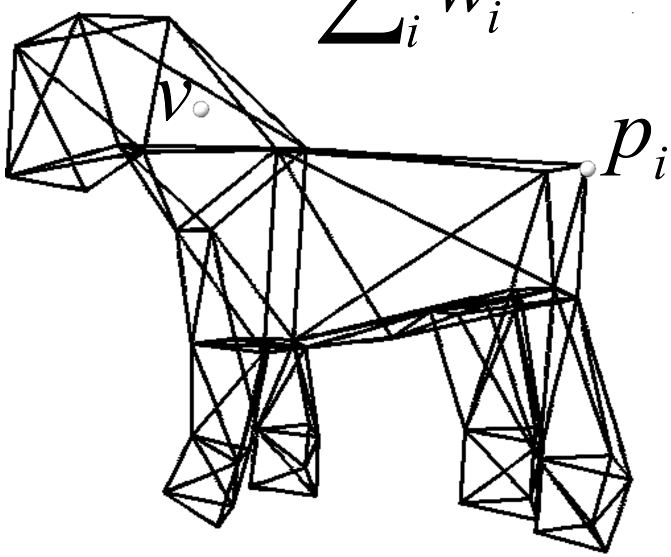
Mean value
(G: unit sphere)



Discrete harmonic
(G: the polyhedron)

Deformations using Barycentric Coordinates

$$v = \frac{\sum_i w_i p_i}{\sum_i w_i}$$



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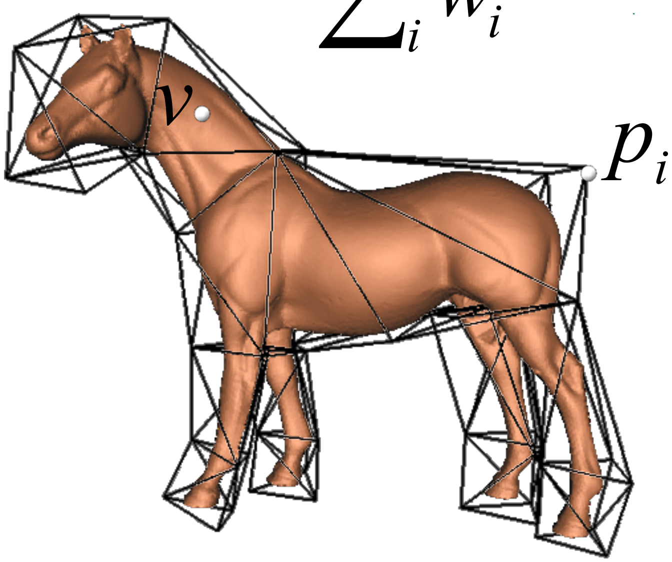

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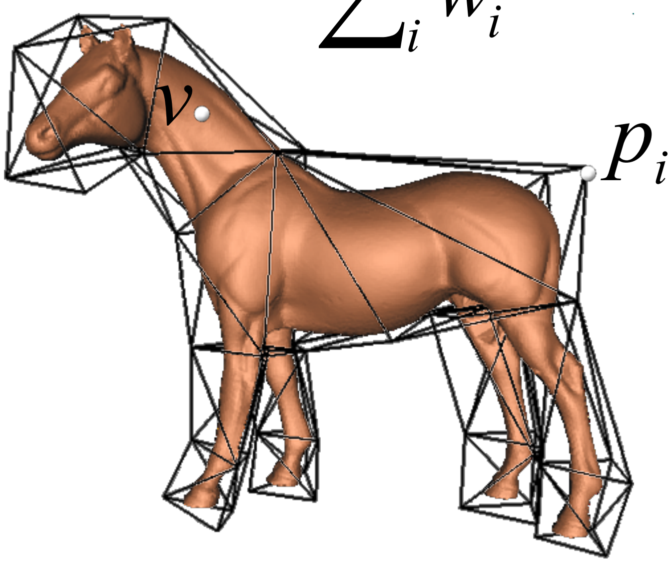
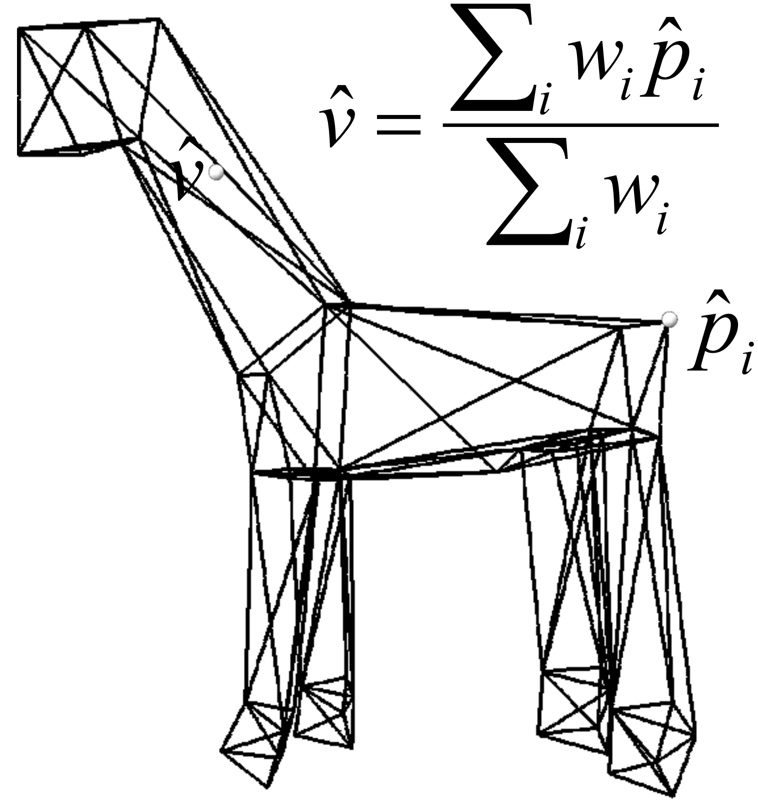


$$\hat{v} = \frac{\sum_i w_i \hat{p}_i}{\sum_i w_i}$$



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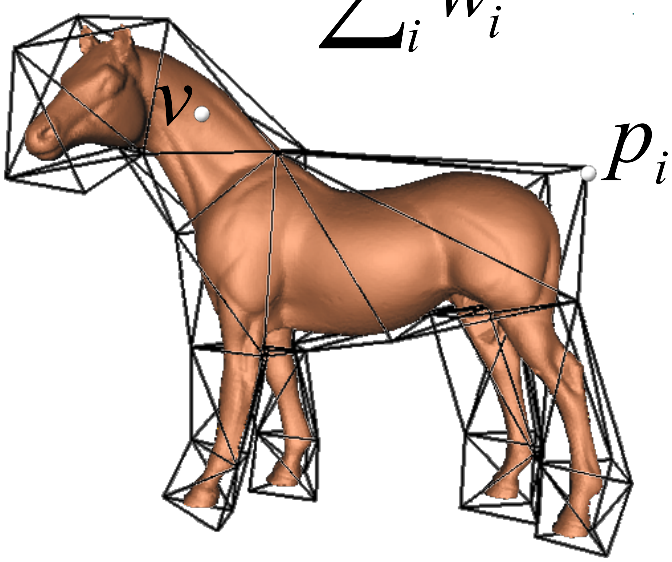

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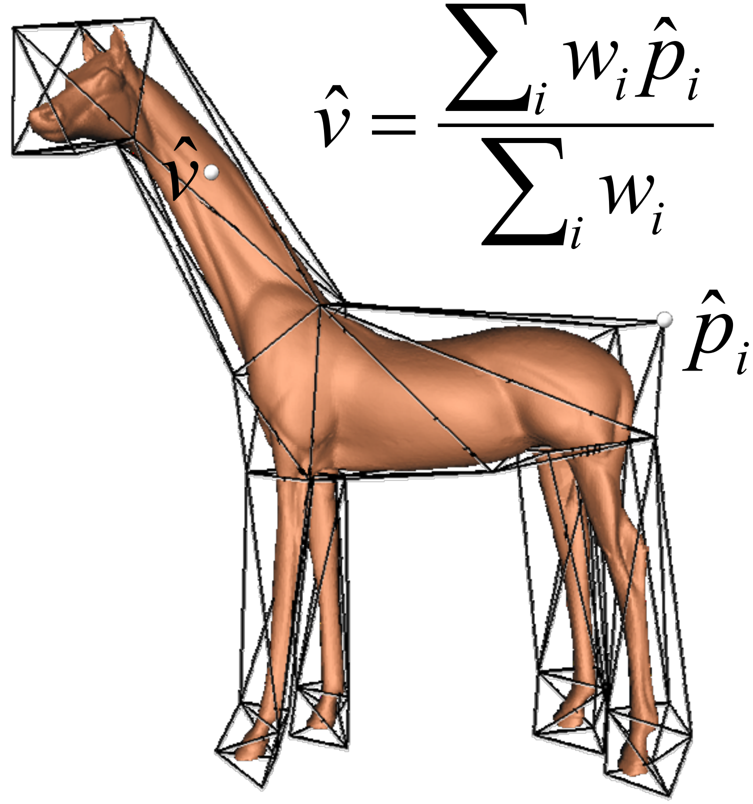
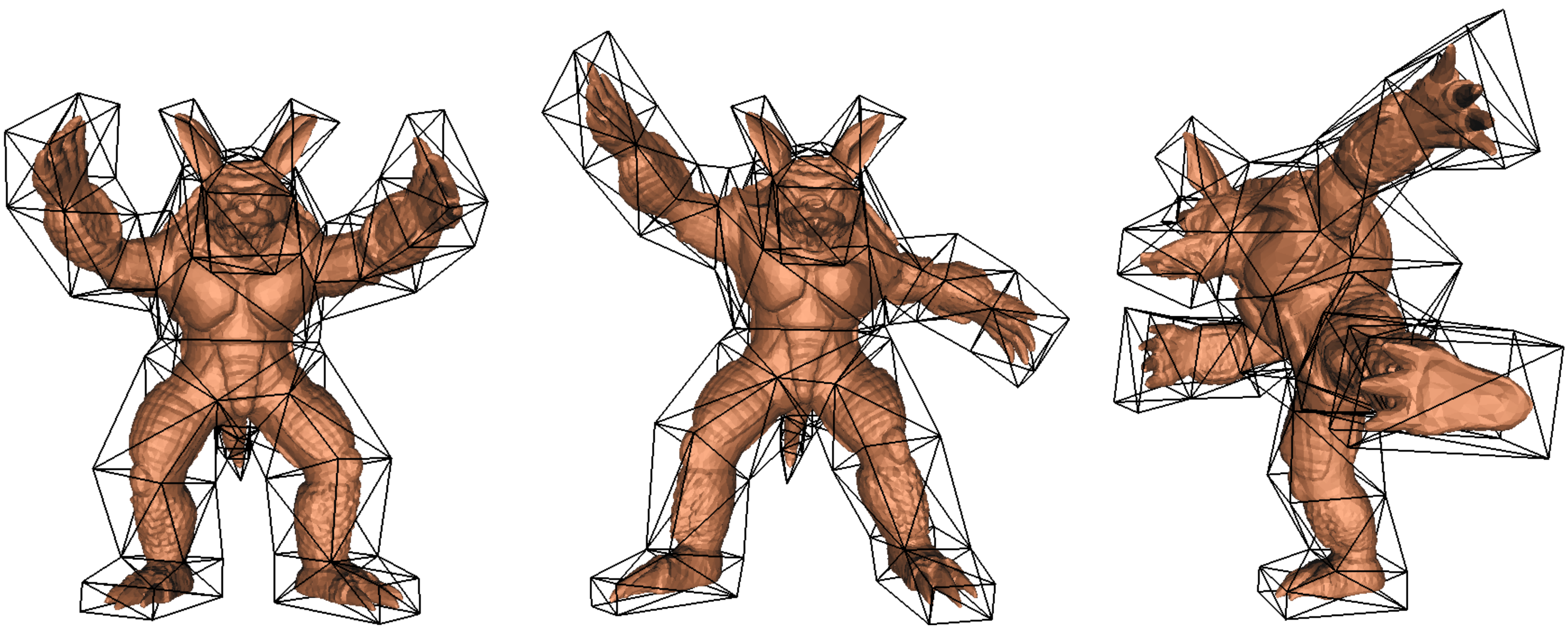


Deformation Examples

Control Mesh	Surface	Computing Weights	Deformation
216 triangles	30,000 triangles	0.7 seconds	0.02 seconds



Deformation Examples

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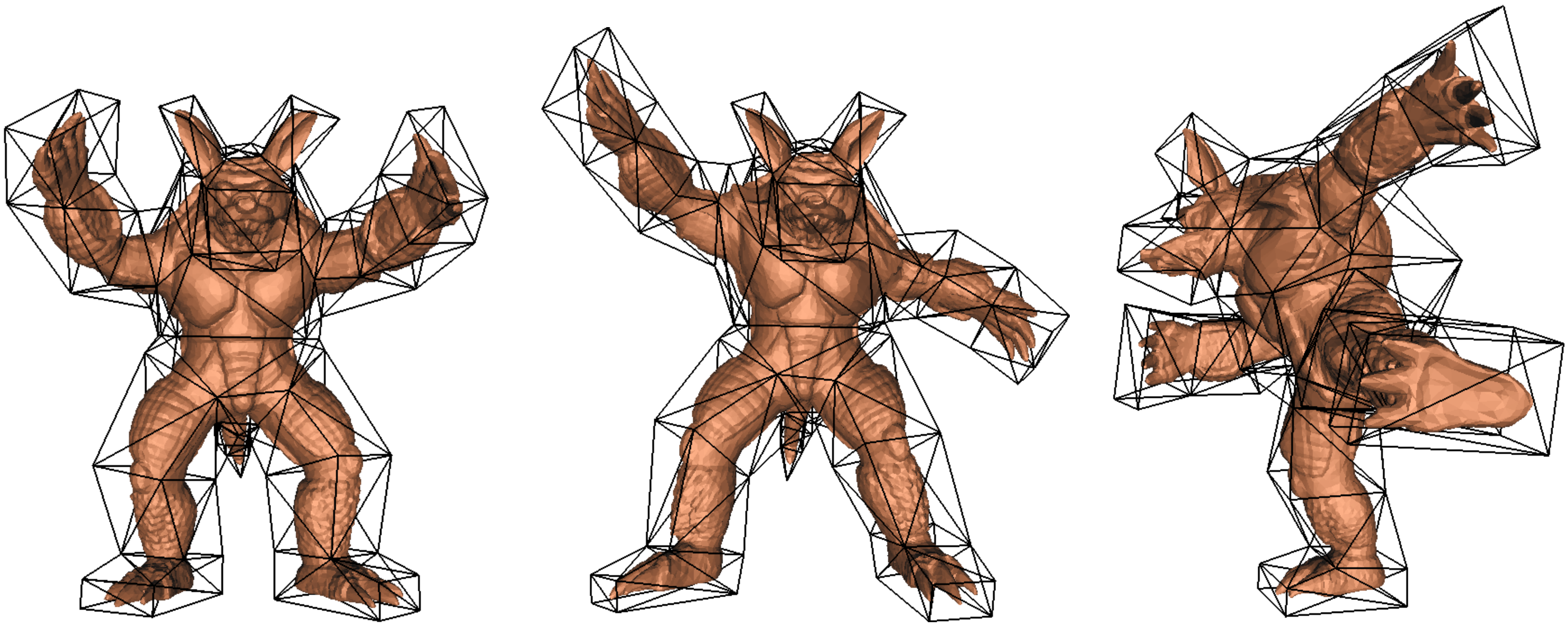
Real-time!



Deformation Examples

Control Mesh	Surface	Computing Weights	Deformation
98 triangles	96,966 triangles	1.1 seconds	0.05 seconds