Linear Algebra
Prerequisites - continued

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Matrices

meaning

$$A \in \Re^{n \times m}$$

n x m matrix

m points from n-dimensional space

transformation

$$C = AA^T$$

Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted) in 2D

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

$$A \in \Re^{2 \times 2}$$

$$\mathbf{y} = A\mathbf{x}$$

Special case
matrix is square

Geometric interpretation

Lines in 2D space - row solution
Equations are considered isolation

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

Linear combination of vectors in 2D
Column solution

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We already know how to multiply the vector by scalar

Linear equations

In 3D

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{y}$$

$$det(A) \neq 0$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$$

$$\mathbf{x} = A^{-1}\mathbf{y}$$

Solving linear n equations with n unknows

If matrix is invertible - compute the inverse

Columns are linearly independent

Linear equations

Not all matrices are invertible

- inverse of a 2x2 matrix (determinant non-zero)
-inverse of a diagonal matrix

Computing inverse - solve for the columns
Independently or using Gauss-Jordan method

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector spaces (informally)

- Vector space in n-dimensional space $\Re^n$
- n-dimensional columns with real entries
- Operations of addition, multiplication and scalar multiplication
- Additions of the vectors and multiplication of a vector by a scalar always produces vectors which lie in the space

- Matrices also make up vector space - e.g. consider all 3x3 matrices as elements of $\Re^9$ space

Vector subspace

A subspace of a vector space is a non-empty set
Of vectors closed under vector addition and scalar multiplication

Example: overconstrained system - more equations then unknowns

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The solution exists if b is in the subspace spanned by vectors u and v

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} x_1 + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear Systems - Nullspace

1. When matrix is square and invertible
2. When the matrix is square and noninvertible
3. When the matrix is non-square with more constraints then unknowns

$$A\mathbf{x} = \mathbf{b}$$

Solution exists when b is in column space of *A*
Special case

All the vectors which satisfy $A\mathbf{x} = 0$ lie in the NULLSPACE of matrix *A*

Basis

n x n matrix A is invertible if it is of a full rank

Rank of the matrix - number of linearly independent rows (see definition next page)

If the rows of columns of the matrix A are linearly independent - the nullspace of contains only 0 vector

Set of linearly independent vectors forms a basis of the vector space
Given a basis, the representation of every vector is unique
Basis is not unique (examples)

Linear independence

Definition A.1 (A linear space). *A set (of vectors) V is considered as* a linear space *over the field $\mathbb{R}$, if its elements, so-called* vectors, *are closed under two basic operations: scalar multiplication and vector summation. That is, given any two vectors $v_1, v_2 \in V$ and any two scalars $\alpha, \beta \in \mathbb{R}$, the linear combination $v = \alpha v_1 + \beta v_2$ is also a vector in V.*

Definition A.4 (Linearly independence). *A set of vectors $S = \{v_i\}_{i=1}^m$ is said to be* linearly independent *if the equation*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Definition A.5 (Basis). *A set of vectors $B = \{b_i\}_{i=1}^n$ of a linear space V is said to be* a basis *if B is a linearly independent set and B spans the entire space V (i.e. $V = span(B)$).*

Change of basis

Fact A.6 (Properties of basis). *Suppose B and B' are two bases for a linear space V. Then*

2. *Let $B = \{b_i\}_{i=1}^n$ and $B' = \{b'_i\}_{i=1}^n$, then each base vector of B can be expressed as linear combination of those in B', i.e.*

$$b_j = a_{1j}b'_1 + a_{2j}b'_2 + \cdots + a_{nj}b'_n = \sum_{i=1}^{n} a_{ij}b'_i. \qquad \text{(A.2)}$$

for some $a_{ij} \in \mathbb{R}, i, j = 1, 2, \ldots, n$.

3. *For any vector $v \in V$, it can be written as a linear combination of vectors in either of the bases*

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n = x'_1 b'_1 + x'_2 b'_2 + \cdots + x'_n b'_n \qquad \text{(A.3)}$$

where the coefficients $\{x_i \in \mathbb{R}\}_{i=1}^n$ and $\{x'_i \in \mathbb{R}\}_{i=1}^n$ are uniquely determined and are called the coordinates of v with respect to each basis.

Change of basis (contd.)

$$[b_1, b_2, \ldots, b_n] = [b'_1, b'_2, \ldots, b'_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$$v = [b_1, b_2, \ldots, b_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [b'_1, b'_2, \ldots, b'_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{B' = BA^{-1}, \quad x' = Ax.}$$

Linear Equations

Vector space spanned by columns of *A* $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of *A* – dimension of C(*A*)
  number of linearly independent columns
  r = rank(*A*)
- Row space of *A* - dimension of R(*A*)
  number of linearly independent rows
  r = rank(A^T)
- Null space of *A* - dimension of N(*A*) n - r
- Left null space of *A* – dimension of N(*A*^T) m – r
- Maximal rank - min(n,m) – smaller of the two dimensions

Linear Equations

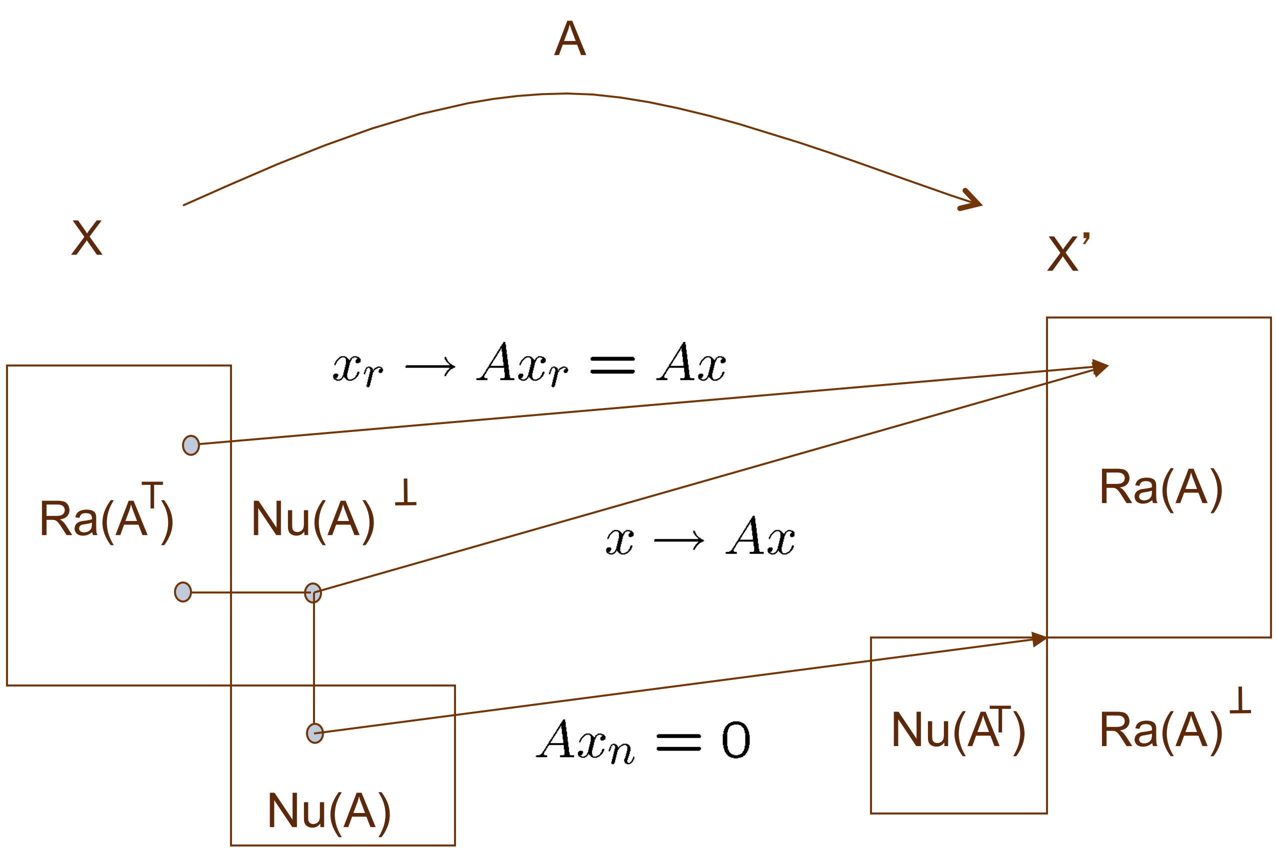
Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic possibilities, suppose that the matrix A has full rank
Then:

- if n < m number of equations is less then number of unknowns, the set of solutions is (m-n) dimensional vector subspace of R^m
- if n = m there is a unique solution
- if n > m number of equations is more then number of unknowns, there is no solution

Structure induced by a linear map



Linear Equations - Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System $Ax = b$ has at most one solution – columns
   are linearly independent rank = n
   - then the matrix is invertible $\mathbf{x} = A^{-1}\mathbf{y}$
2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations - non-square matrices

Long-tin matrix over-constrained system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation - least squares
Approximation - minimize squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution - find such value of x that the error Is minimized (take a derivative, set it to zero and solve for x)

Short for such solution

$$e^2 = ||ax - b||^2$$

$$a\mathbf{x} = b$$
$$a^T a\mathbf{x} = a^T b$$
$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

Linear equations - non-squared matrices

Similarly when A is a matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$e^2 = \|A\mathbf{x} - b\|^2$$

$$A\mathbf{x} = b$$

$$A^T A\mathbf{x} = A^T b$$

$$\bar{\mathbf{x}} = (A^T A)^{-1} A^T b$$

- If A has linearly independent columns $A^T A$ is square, symmetric and invertible

$$A^\dagger = (A^T A)^{-1} A^T$$

is so called pseudoinverse of matix A

Homogeneous Systems of equations

$$A\mathbf{x} = 0$$

When matrix is square and non-singular, there a Unique trivial solution x = 0

If m >= n there is a non-trivial solution when rank of *A* is rank(*A*) < n
We need to impose some constraint to avoid trivial Solution, for example

$$\|\mathbf{x}\| = 1$$

Find such x that $\|A\mathbf{x}\|^2$ is minimized

$$\|A\mathbf{x}\|^2 = \mathbf{x}A^T A\mathbf{x}$$

Solution: eigenvector associated with the smallest eigenvalue

Eigenvalues and Eigenvectors

- Motivated by solution to differential equations
- For square matrices $A \in \Re^{n \times n}$ $\quad \dot{\mathbf{u}} = A\mathbf{u}$ $\quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

For scalar ODE's

$$\dot{u} = au$$

$$u(t) = e^{at}u(0)$$

We look for the solutions of the following type exponentials

$$v(t) = e^{\lambda t}y$$

$$w(t) = e^{\lambda t}z$$

Substitute back to the equation

$$\lambda \cancel{e^{\lambda t}} y = 4\cancel{e^{\lambda t}} y - 5\cancel{e^{\lambda t}} z$$

$$\lambda \cancel{e^{\lambda t}} z = 2\cancel{e^{\lambda t}} y - 3\cancel{e^{\lambda t}} z$$

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix} \qquad \lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\mathbf{x}$$

Eigenvalues and Eigenvectors

$$\lambda \mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

eigenvalue (λ), eigenvector ($\mathbf{x}$)

Solve the equation: $(A - \lambda I)\mathbf{x} = 0$ (1)

x – is in the null space of $(A - \lambda I)$
λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

Eigenvalues and Eigenvectors

For the previous example

$$\lambda_1 = -1, x_1 = [1,1]^T \qquad \lambda_2 = -2, x_2 = [5,2]^T$$

We will get special solutions to ODE $\dot{\mathbf{u}} = A\mathbf{u}$

$$A\mathbf{u} = e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{u} = e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Their linear combination is also a solution (due to the linearity of $\dot{\mathbf{u}} = A\mathbf{u}$)

$$\mathbf{u} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

In the context of diff. equations – special meaning
Any solution can be expressed as linear combination
Individual solutions correspond to modes

Eigenvalues and Eigenvectors

$$Ax = \lambda x$$

Only special vectors are eigenvectors
- such vectors whose direction will not be changed by the transformation A (only scale)
- they correspond to normal modes of the system act independently

Examples

	eigenvalues	eigenvectors
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	2, 3	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Whatever A does to an arbitrary vector is fully determined by its eigenvalues and eigenvectors

$$Ax = 2\lambda_1 v_1 + 5\lambda_2 v_2$$

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S \Lambda S^{-1}$$

$$A = S \Lambda S^{-1}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \Lambda S$$

$$A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix}$$

$$A\mathbf{x} = \lambda \mathbf{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & \dots & \\ & & & \lambda_n \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

- If some of the eigenvalues are the same, eigenvectors are not independent

Diagonalization

- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

For Symmetric Matrices

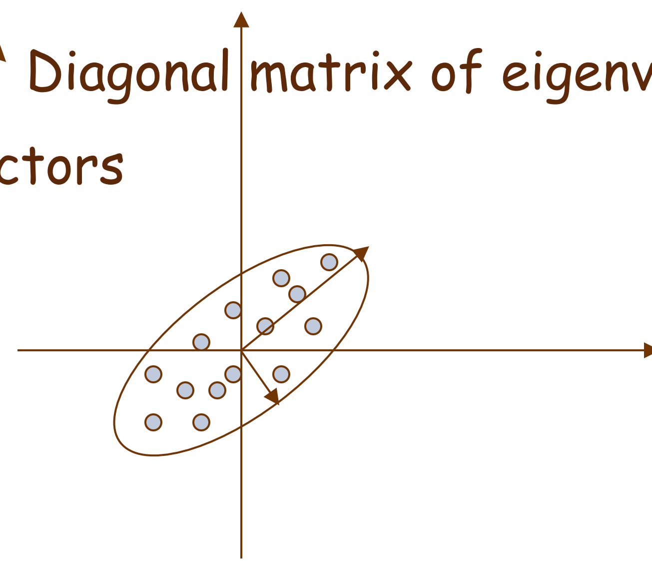
If A is symmetric $$A = Q\Lambda Q^T$$

Λ: Diagonal matrix of eigenvalues

Q: orthonormal matrix of eigenvectors

i.e. for a covariance matrix

or some matrix B = A^TA



Symmetric matrices (contd.)

$$A^T A = V \text{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\} V^T$$

$$\|A\|_f = \sqrt{\sum_{i,j} a_{ij}^2}.$$

$$\|A\|_f \doteq \sqrt{\text{trace}(A^T A)} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.$$

Example - line fitting

Equation of a line $$ax + by = d$$

Line normal $$\mathbf{n} = [a, b]$$

Distance to the origin d

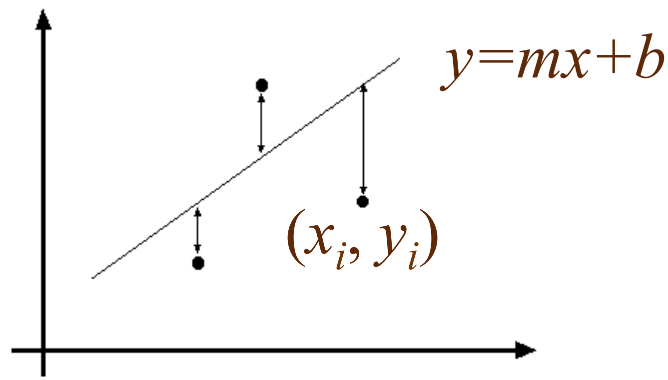
Error function $$e(a, b, d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

Differentiate with respect to a,b,d
set the first derivative to 0 and solve for the parameters

Least squares line fitting

- Data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Line equation: $y_i = m\,x_i + b$
- Find (m, b) to minimize

$$E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$



$$\vec{b} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \left\| \vec{b} - A\vec{x} \right\|^2 = (\vec{b} - A\vec{x})^T (\vec{b} - A\vec{x}) = \vec{b}^T \vec{b} - 2(A\vec{x})^T \vec{b} + (A\vec{x})^T (A\vec{x})$$

$$\frac{dE}{d\vec{x}} = 2A^T A\vec{x} - 2A^T \vec{b} = 0$$

$$A^T A\vec{x} = A^T \vec{b}$$

Normal equations: least squares solution to $A\vec{x} = \vec{b}$

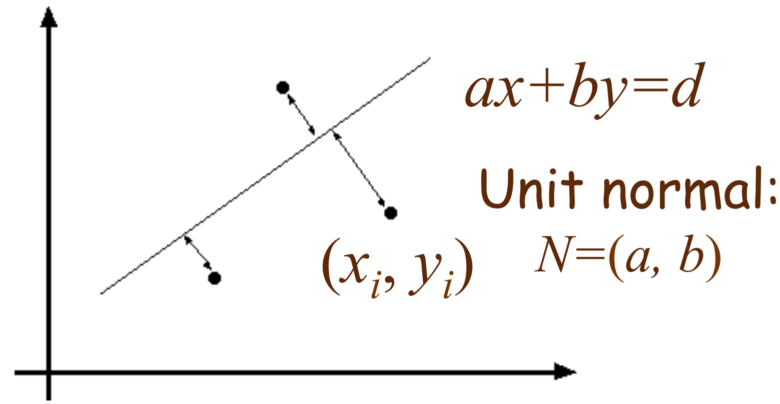
Problem with "vertical" least squares

- Not rotation-invariant
- Fails completely for vertical lines

Total least squares

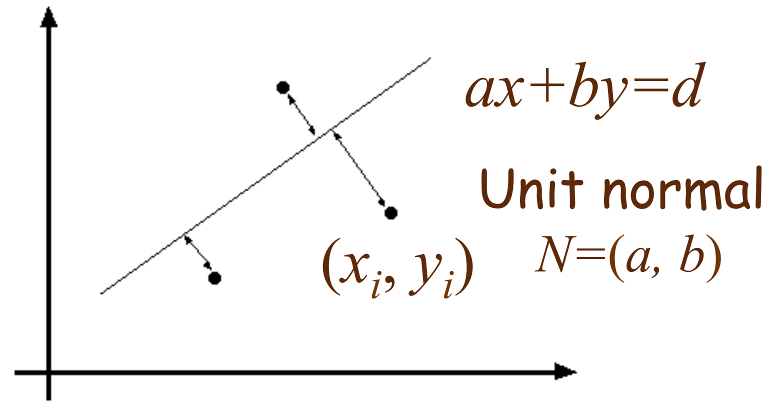
- Distance between point
- (x_i, y_i) and line $ax+by=d$

$(a^2+b^2=1)$: $|ax_i + by_i - d|$



Total least squares

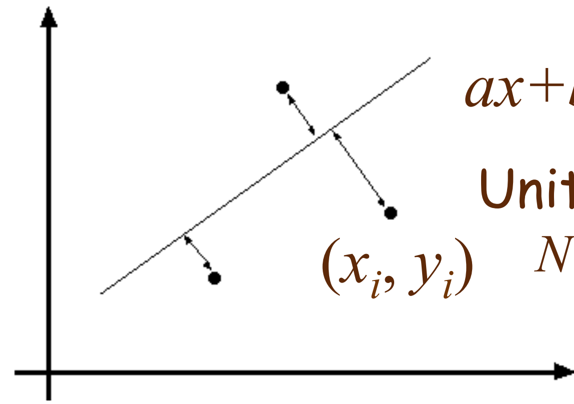
- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$E = \sum_{i=1}^{n}(ax_i + by_i - d)^2$$

Total least squares

- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$E = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0 \qquad d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\bar{x} + b\bar{y}$$

$$E = \sum_{i=1}^{n} (a(x_i - \bar{x}) + b(y_i - \bar{y}))^2 = \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = (A\vec{u})^T (A\vec{u})$$

$$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \frac{dE}{d\vec{u}} = 2(A^T A)\vec{u} = 0$$

Total least squares

Solution to $(A^T A)\boldsymbol{u} = 0$, subject to $\|\boldsymbol{u}\|^2 = 1$: eigenvector of $A^T A$ associated with the smallest eigenvalue (least squares solution to *homogeneous linear system* $A\vec{u} = 0$)

In case of 2D line fitting

$$A = \begin{bmatrix} x_1 - \overline{x} & y_1 - \overline{y} \\ \vdots & \vdots \\ x_n - \overline{x} & y_n - \overline{y} \end{bmatrix} \qquad A^T A = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \overline{x})^2 & \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y}) \\ \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y}) & \sum_{i=1}^{n} (y_i - \overline{y})^2 \end{bmatrix}$$

second moment matrix - geometric interpretation of eigenvalues and eigenvectors

$$A^T A = \begin{bmatrix} \sum_{i=1}^{n}(x_i - \overline{x})^2 & \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) \\ \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) & \sum_{i=1}^{n}(y_i - \overline{y})^2 \end{bmatrix}$$

second moment matrix