CS 687
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Inference in Bayesian Networks

Chapter 14, Russell and Norvig

Bayesian Networks

- Syntax
- Semantics

Previously: Bayes' Rule

- Bayes Rule

$$P(Y \mid X) = \frac{P(X \mid Y)P(Y)}{P(X)}$$

- Useful for assessing diagnostic probability from causal probability:
  P(Cause|Effect) = P(Effect|Cause) P(Cause) / P(Effect)

P(Y)

P(X|Y)

X is observed, we want P(Y | X)

Modeling joint distribution as product of conditional and prior

P(X,Y) = P(X|Y)P(Y)

Bayesian networks

- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

- Syntax:
  - a set of nodes, one per variable

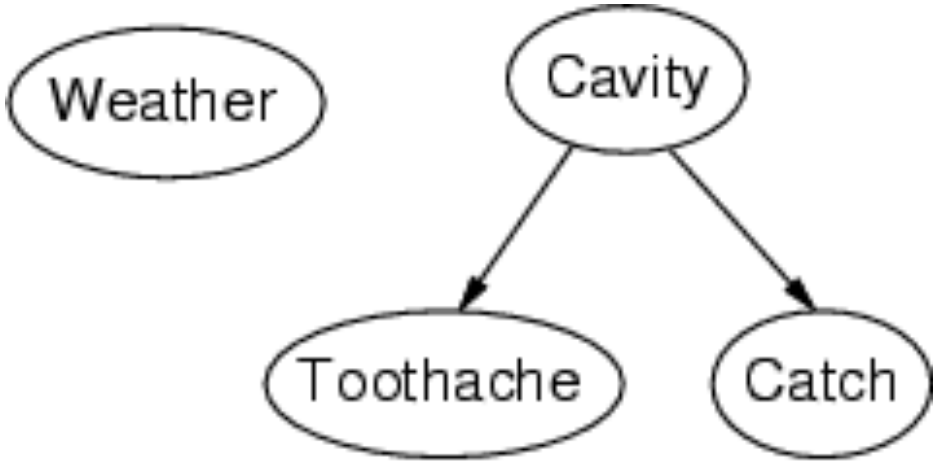
  - a directed, acyclic graph (link ≈ "directly influences")
  - a conditional distribution for each node given its parents:

    $$\mathbf{P}(X_i \mid \text{Parents}(X_i))$$

- In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

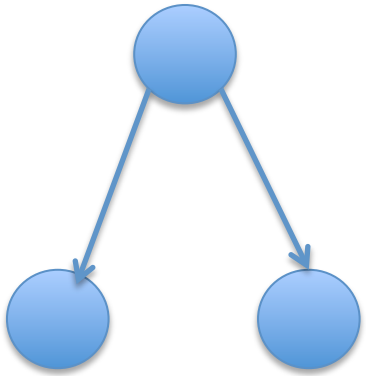
Example

- Topology of network encodes conditional independence assertions:

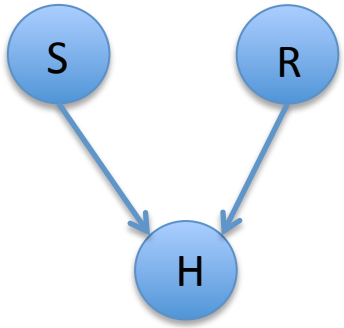


- *Weather* is independent of the other variables
- *Toothache* and *Catch* are conditionally independent given *Cavity*

Different types of BN reasoning



Observe evidence
Reason about cause



Explaining Away

P(S) = 0.7
P(R) = 0.01
P(H | S, R) = 1
P(H | ~S, R) = 0.9
P(H | S, ~R) = 0.7
P(H | ~S, ~R) = 0.1

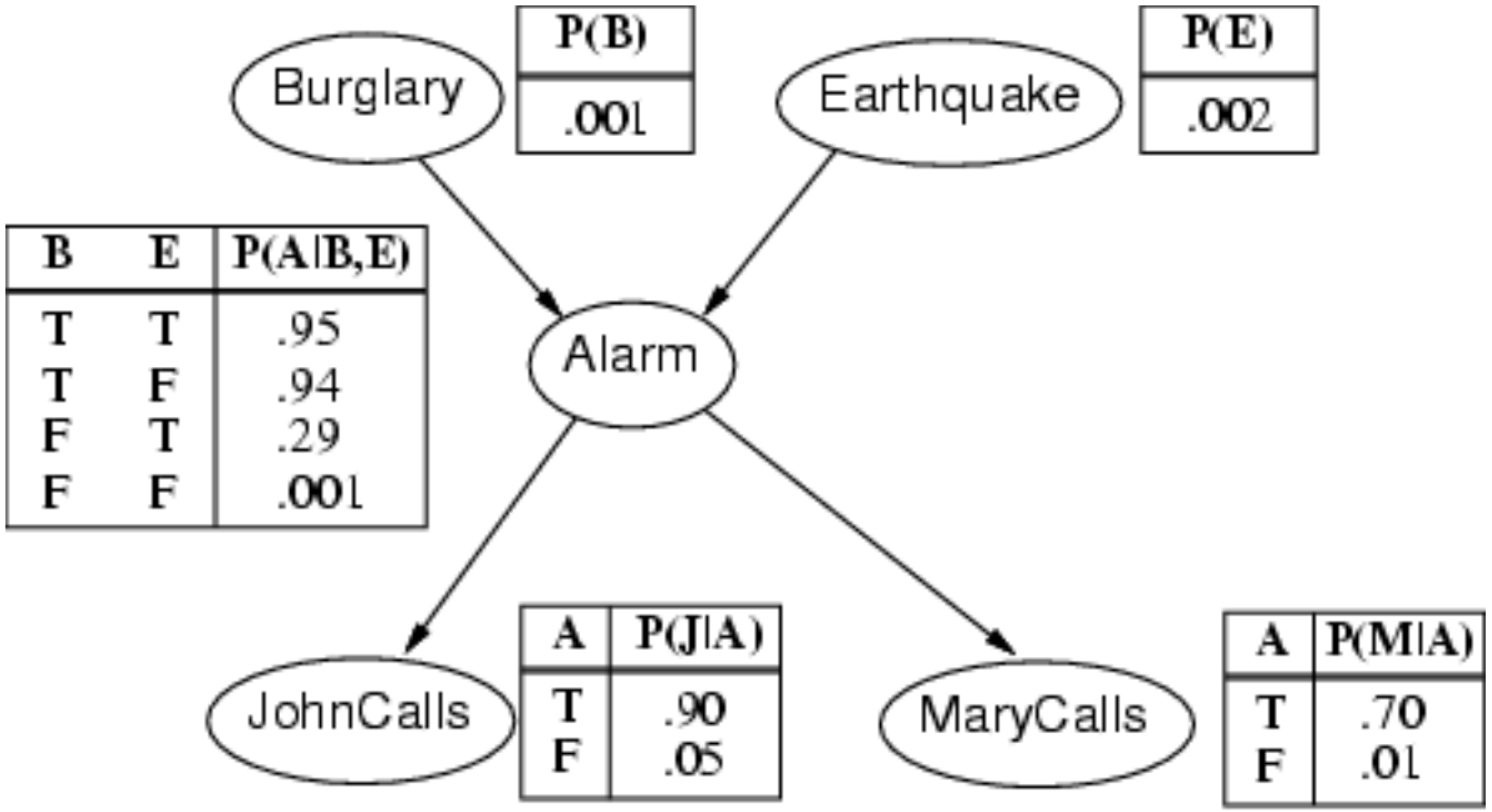
Example

- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

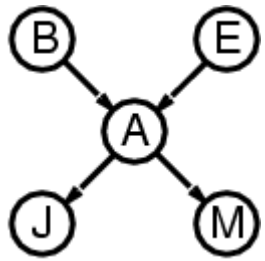
- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

Example contd.



Compactness

- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values



- Each row requires one number p for X_i = *true* (the number for X_i = *false* is just *1-p*)

- If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

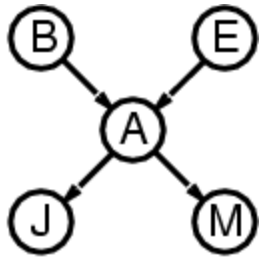
- I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

- For burglary net, 1 + 1 + 4 + 2 + 2 = 10 numbers (vs. 2^5-1 = 31)

Semantics

The full joint distribution is defined as the product of the local conditional distributions:

$$\mathbf{P}(X_1, \ldots, X_n) = \pi_{i=1}^{n} \mathbf{P}(X_i \mid \text{Parents}(X_i))$$



e.g., $\mathbf{P}(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$= \mathbf{P}(j \mid a)\, \mathbf{P}(m \mid a)\, \mathbf{P}(a \mid \neg b, \neg e)\, \mathbf{P}(\neg b)\, \mathbf{P}(\neg e)$

Local Semantics

- Each node is conditionally independent of its non-descendants given its parents
- Each node is conditionally independent of all others given its Markov Blanket: parents + children+children's parents

- More general notion of conditional independence
- d-separation
- Helps decide whether set of nodes X is independent from set on nodes Y, given third set

Reasoning Patterns in Bayes Nets

- Blackboard
- Causal reasoning – downstream; evidence are causes – compute probabilities of effects
- Evidential reasoning – observe effects, compute probability of causes
- Inter-causal reasoning – Explaining away: different causes of the same effect can interact: explain each other away (flu, mono example)

- Cancer example, happiness example

Outline

- Exact inference by enumeration
- Exact inference by variable elimination
- Approximate inference by stochastic simulation
- Approximate inference by Markov Chain Monte Carlo

Inference by enumeration

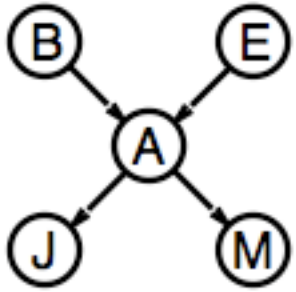
- Single chain example (black-board)

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\&=\mathbf{P}(B,j,m)/P(j,m)\\&=\alpha\mathbf{P}(B,j,m)\\&=\alpha\sum_e\sum_a\mathbf{P}(B,e,a,j,m)\end{aligned}$$



Rewrite full joint entries using product of CPT entries:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\&=\alpha\sum_e\sum_a\mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)\\&=\alpha\mathbf{P}(B)\sum_e P(e)\sum_a\mathbf{P}(a|B,e)P(j|a)P(m|a)\end{aligned}$$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Inference by enumeration

- The number of operations for inference by enumeration depends on how is network constructed

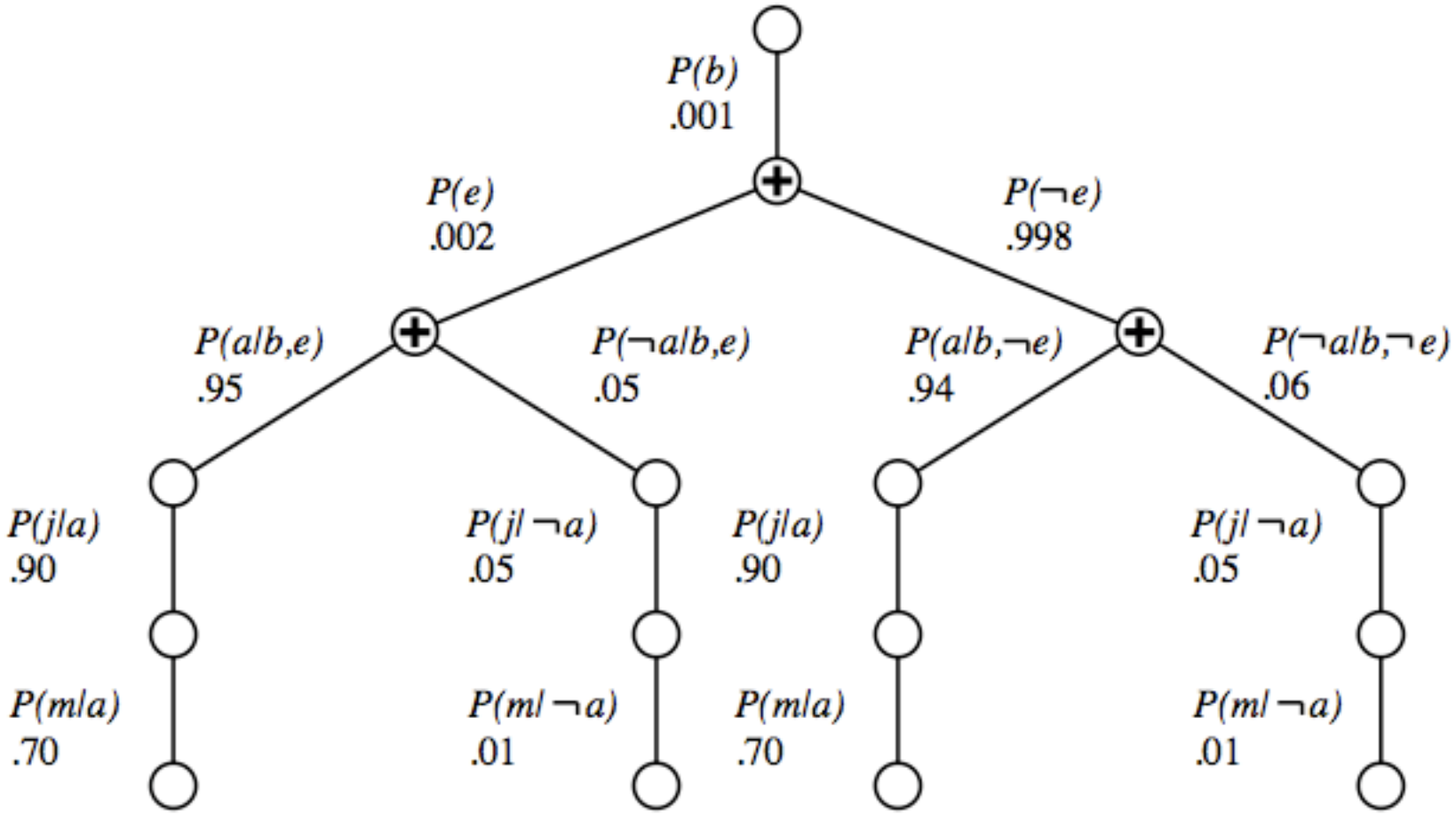
Enumeration Algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    extend $\mathbf{e}$ with value x_i for X
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[bn], $\mathbf{e}$)
  return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\Sigma_y \; P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Evaluation Tree



Enumeration is inefficient: repeated computation

e.g., computes $P(j|a)P(m|a)$ for each value of e

Constructing Bayesian networks

- 1. Choose an ordering of variables $X_1, \ldots, X_n$
- 2. For $i = 1$ to n
  - add X_i to the network
  - select parents from $X_1, \ldots, X_{i-1}$ such that

$$\boldsymbol{P}(X_i \mid Parents(X_i)) = \boldsymbol{P}(X_i \mid X_1, \ldots X_{i-1})$$

This choice of parents guarantees:

$$\boldsymbol{P}(X_1, \ldots, X_n) = \pi_{i=1}^{n} \boldsymbol{P}(X_i \mid X_1, \ldots, X_{i-1}) \quad \text{(chain rule)}$$

$$= \pi_{i=1}^{n} \boldsymbol{P}(X_i \mid Parents(X_i)) \quad \text{(by construction)}$$

Example

- Suppose we choose the ordering *M, J, A, B, E*

***P**(J | M) = **P**(J)?*

Example

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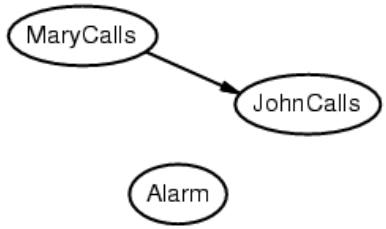

***P**(J | M) = **P**(J)?*

No

***P**(A | J, M) = **P**(A | J)? **P**(A | J, M) = **P**(A)?*

Example

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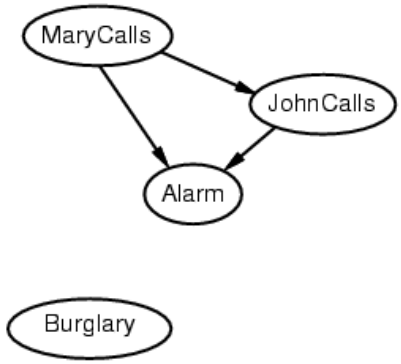


***P**(J | M) = **P**(J)?*

No

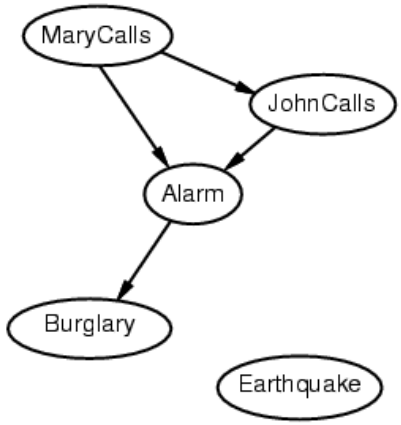
***P**(A | J, M) = **P**(A | J)? **P**(A | J, M) = **P**(A)?* **No**

***P**(B | A, J, M) = **P**(B | A)?*

***P**(B | A, J, M) = **P**(B)?*

Example

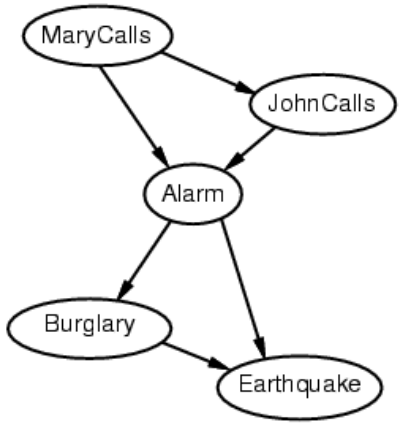
- Suppose we choose the ordering M, J, A, B, E



P(*J* | *M*) = ***P***(*J*)?
No
P(*A* | *J*, *M*) = ***P***(*A* | *J*)? ***P***(*A* | *J*, *M*) = ***P***(*A*)? **No**
P(*B* | *A*, *J*, *M*) = ***P***(*B* | *A*)? **Yes**
P(*B* | *A*, *J*, *M*) = ***P***(*B*)? **No**
P(*E* | *B*, *A* ,*J*, *M*) = ***P***(*E* | *A*)?
P(*E* | *B*, *A*, *J*, *M*) = ***P***(*E* | *A*, *B*)?

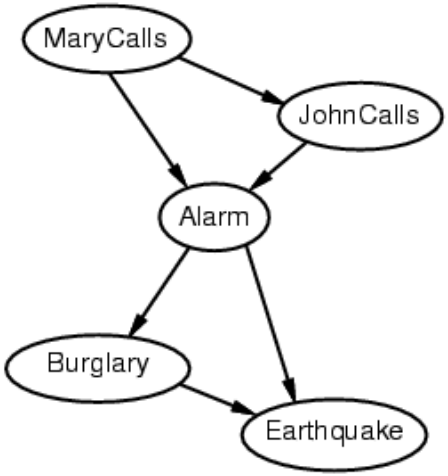
Example

- Suppose we choose the ordering M, J, A, B, E



P(*J* | *M*) = ***P***(*J*)?
No
P(*A* | *J*, *M*) = ***P***(*A* | *J*)? ***P***(*A* | *J*, *M*) = ***P***(*A*)? **No**
P(*B* | *A*, *J*, *M*) = ***P***(*B* | *A*)? **Yes**
P(*B* | *A*, *J*, *M*) = ***P***(*B*)? **No**
P(*E* | *B*, *A* ,*J*, *M*) = ***P***(*E* | *A*)? **No**
P(*E* | *B*, *A*, *J*, *M*) = ***P***(*E* | *A*, *B*)? **Yes**

Example contd.



- Deciding conditional independence is hard in noncausal directions
- (Causal models and conditional independence seem hardwired for humans!)
- Network is less compact: 1 + 2 + 4 + 2 + 4 = 13 numbers needed

Inference by variable elimination

Variable elimination: carry out summations right-to-left, storing intermediate results (factors) to avoid recomputation

$$
\begin{aligned}
&\mathbf{P}(B|j,m) \\
&\quad = \alpha \underbrace{\mathbf{P}(B)}_{B} \Sigma_e \underbrace{P(e)}_{E} \Sigma_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M} \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a f_A(a,b,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)} \\
&\quad = \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)} \\
&\quad = \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Variable elimination basic operations

Summing out a variable from a product of factors:

move any constant factors outside the summation

add up submatrices in pointwise product of remaining factors

$$\sum_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \sum_x f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

$$f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l)$$
$$= f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l)$$

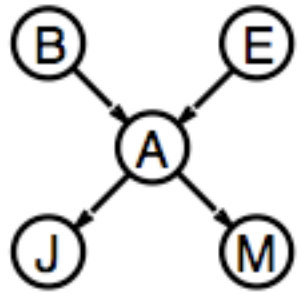
E.g., $f_1(a, b) \times f_2(b, c) = f(a, b, c)$

Irrelevant variables

Consider the query $P(JohnCalls|Burglary=true)$

$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over m is identically 1; M is **irrelevant** to the query



Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so $MaryCalls$ is irrelevant

(Compare this to backward chaining from the query in Horn clause KBs)

Complexity of exact inference

Singly connected networks (or polytrees):

- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:

- can reduce 3SAT to exact inference $\Rightarrow$ NP-hard
- equivalent to **counting** 3SAT models $\Rightarrow$ #P-complete

Variable elimination – inference is linear in the size of the network
(size of CPT and number of nodes

Inference by stochastic simulation

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P

Outline:

- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Inference by Sampling

- Examples (blackboard)
- Estimate joint distribution of two coins (sampling from empty network – no evidence)

- Direct Sampling
- Rejection Sampling
- Importance Sampling
- MCMC

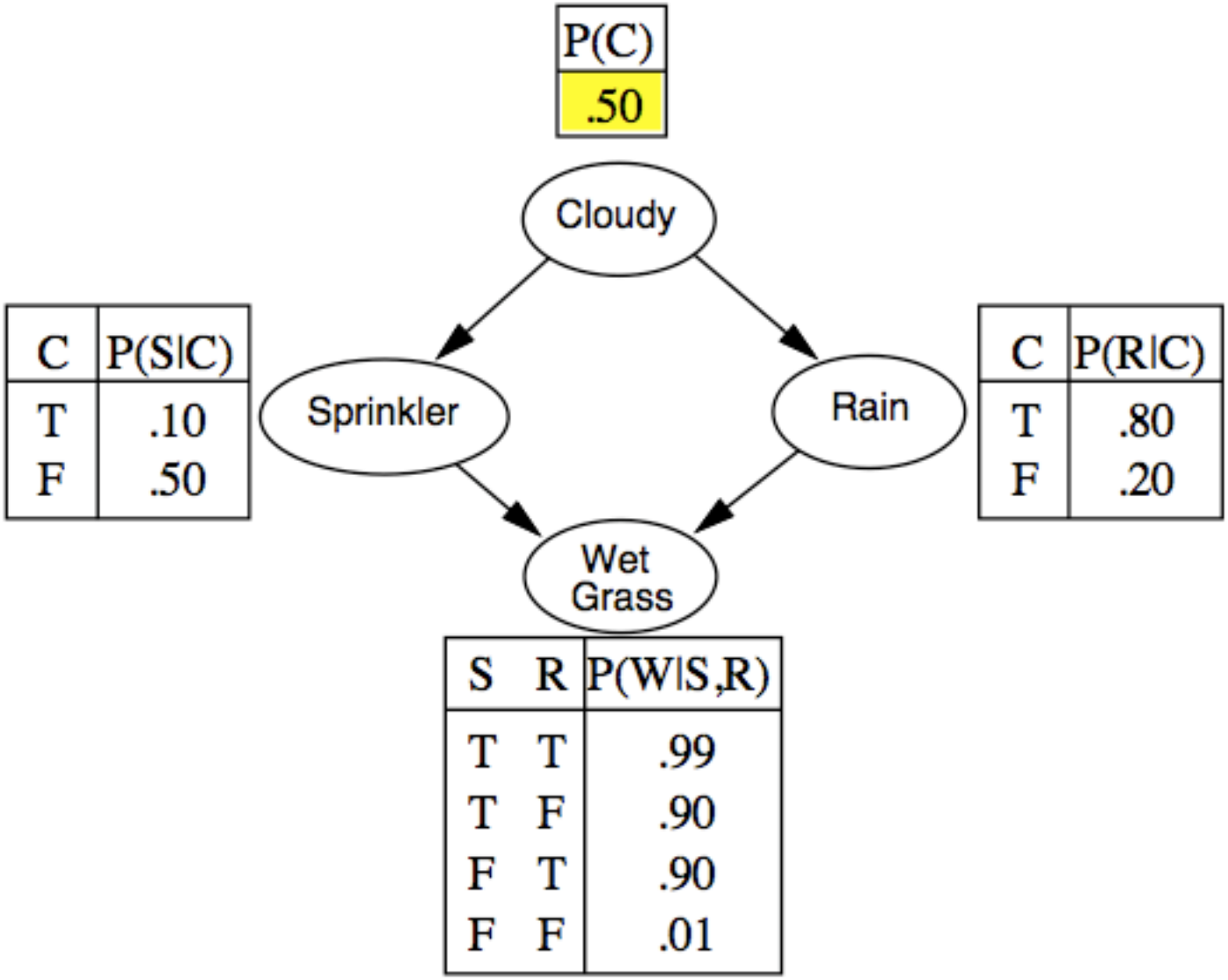
- Sampling -> approximate inference method

Inference by stochastic simulation

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in $\mathbf{x}$
  return $\mathbf{x}$

Example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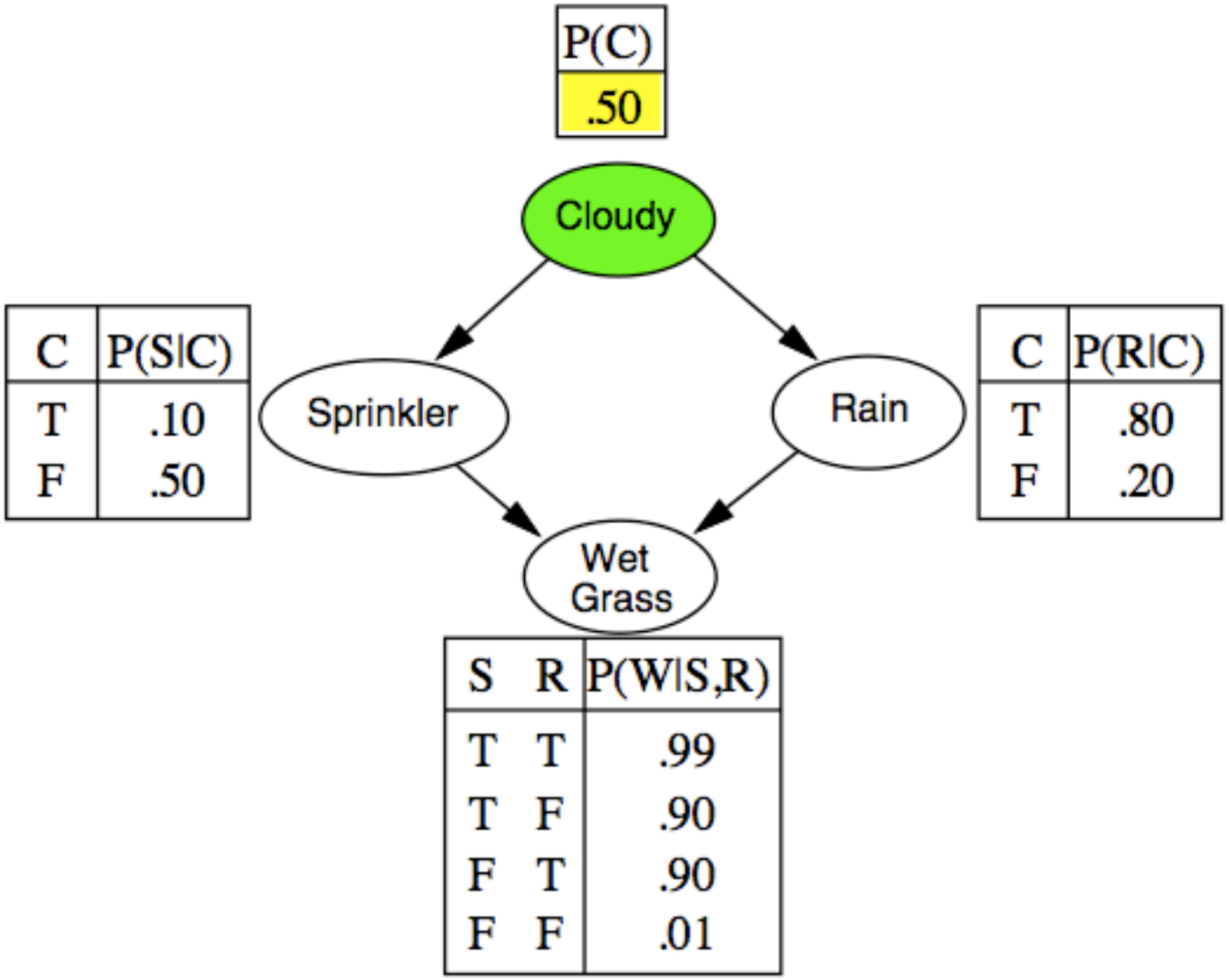
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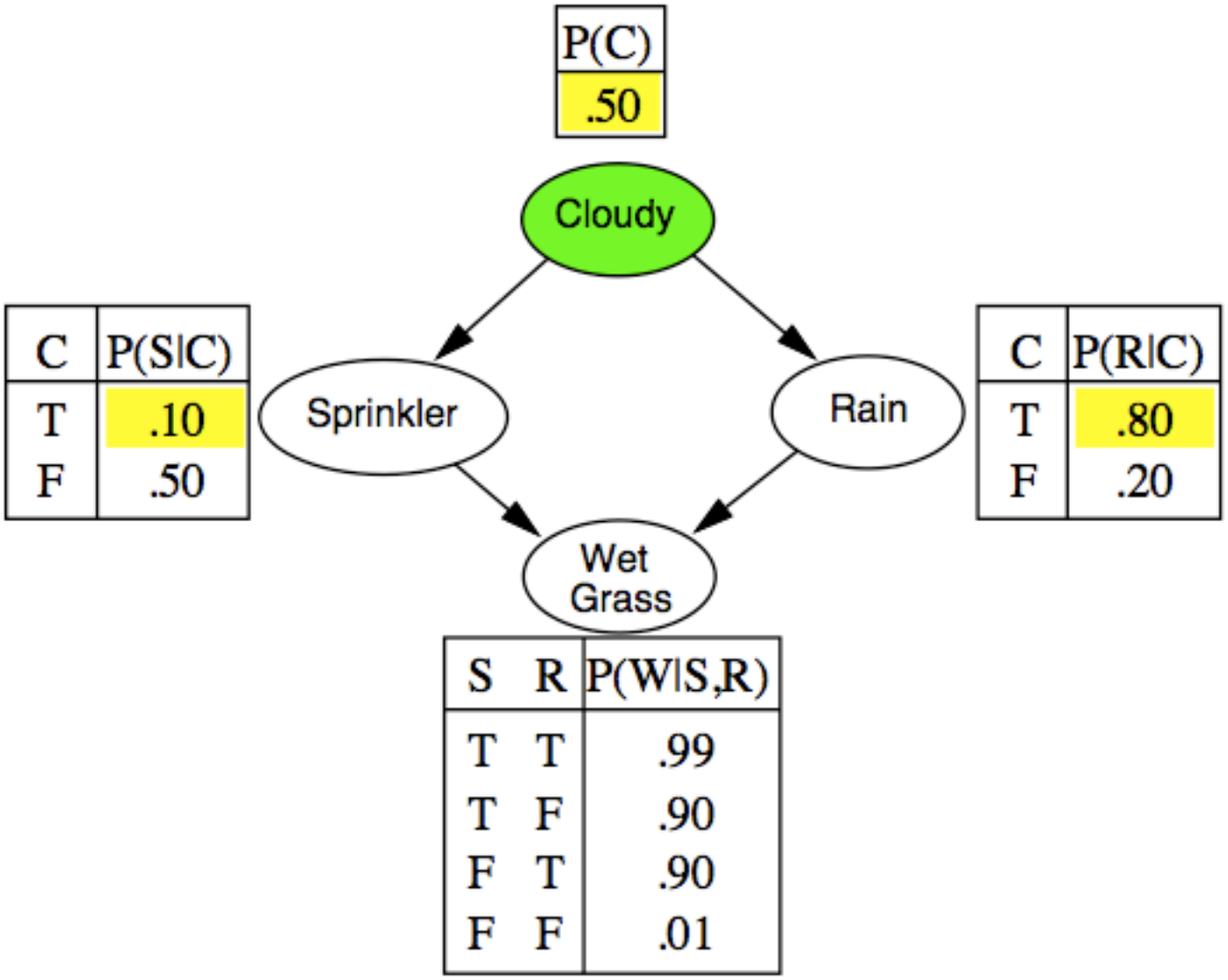
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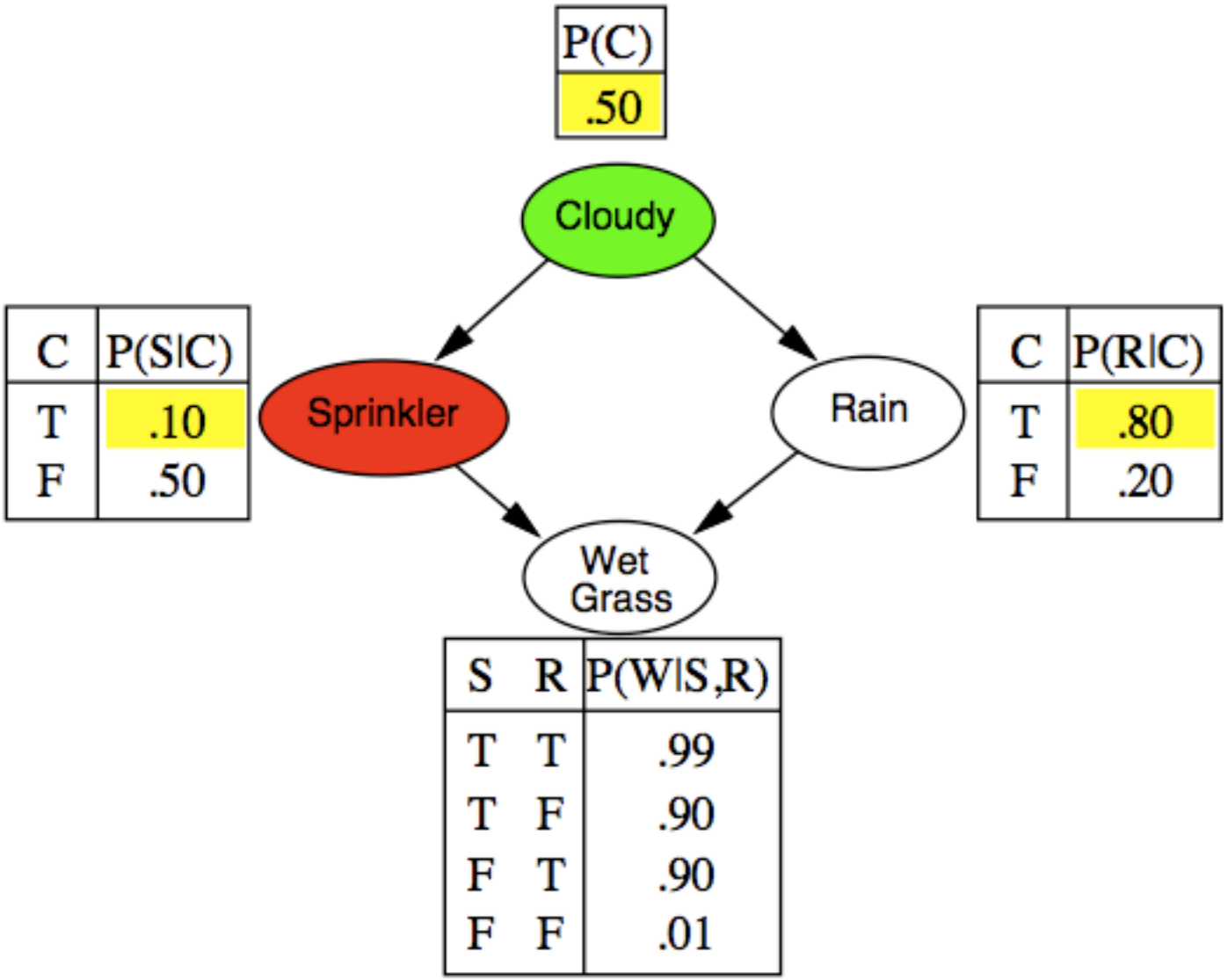
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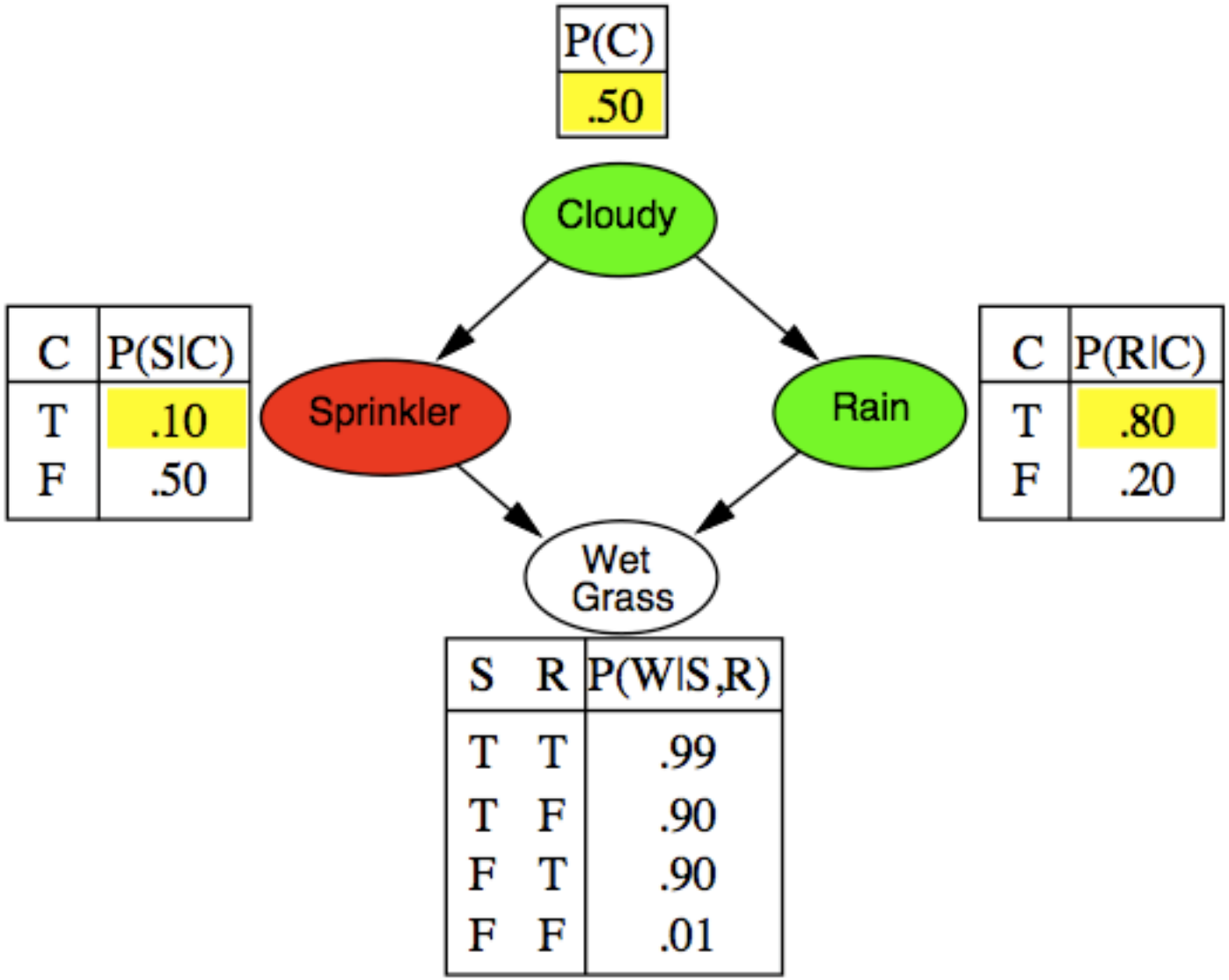
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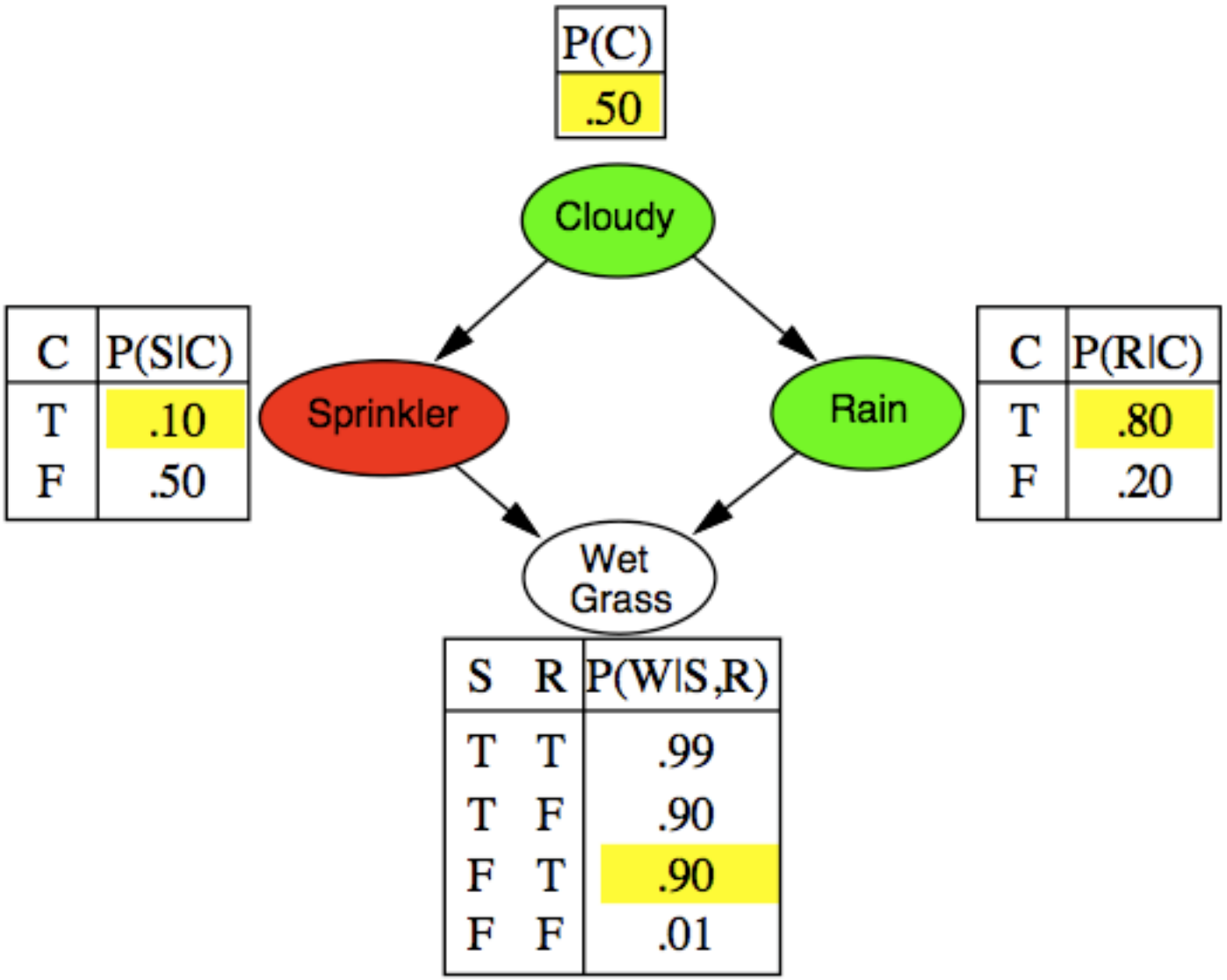
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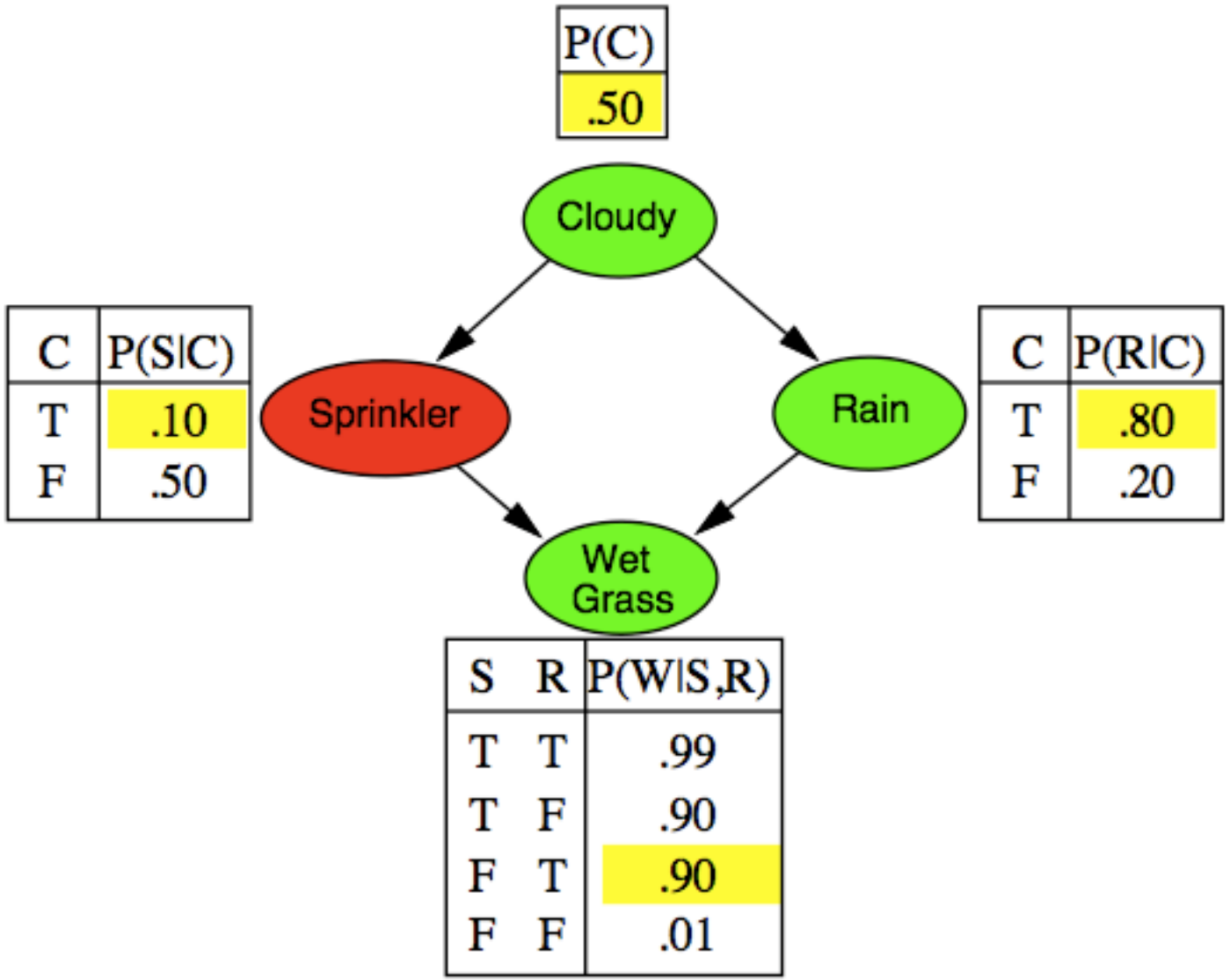
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Sampling from empty network

Probability that PRIORSAMPLE generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
  local variables: N, a vector of counts over X, initially zero

  for j = 1 to N do
      x ← PRIOR-SAMPLE(bn)
      if x is consistent with e then
          N[x] ← N[x]+1 where x is the value of X in x
  return NORMALIZE(N[X])
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples

27 samples have $Sprinkler = true$

Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$$\begin{aligned}
\hat{\mathbf{P}}(X|\mathbf{e}) &= \alpha \mathbf{N}_{PS}(X,\mathbf{e}) && \text{(algorithm defn.)} \\
&= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e}) && \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
&\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e}) && \text{(property of PriorSample)} \\
&= \mathbf{P}(X|\mathbf{e}) && \text{(defn. of conditional probability)}
\end{aligned}$$

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

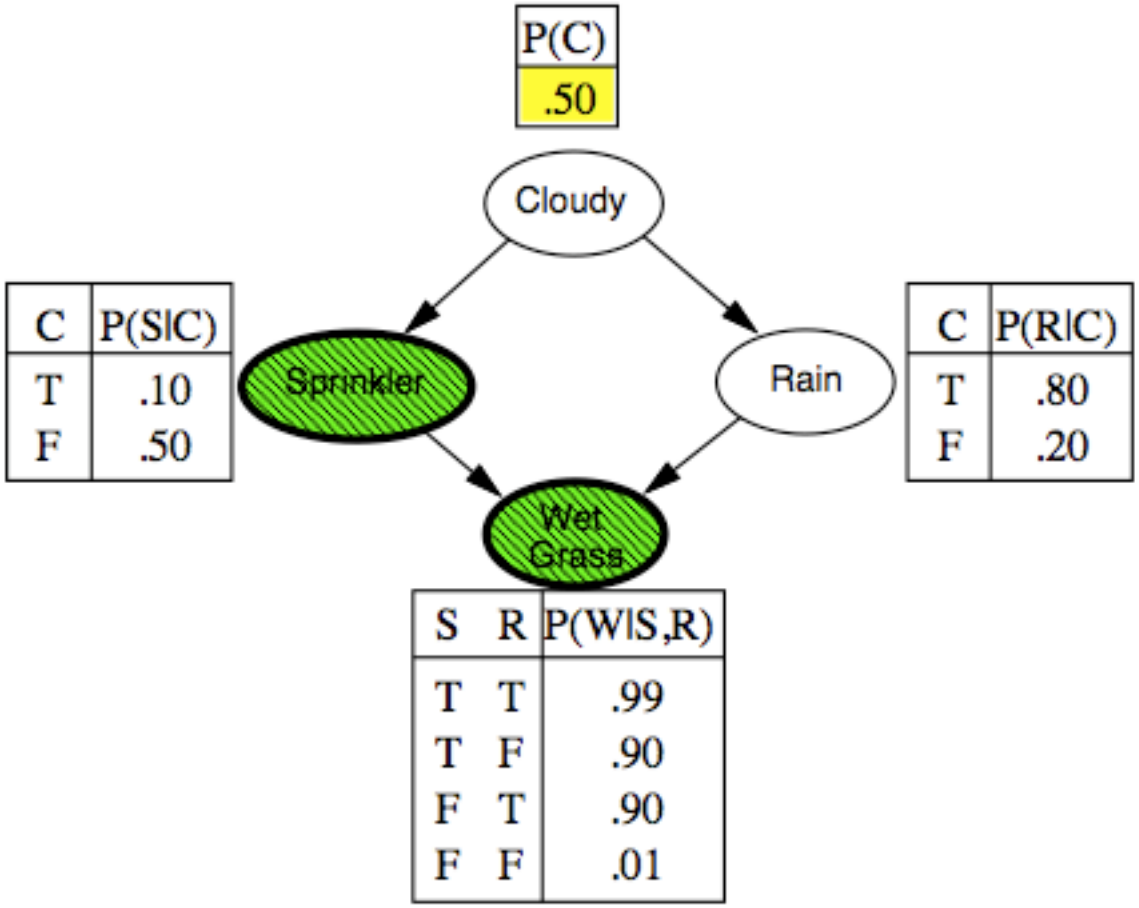
function LIKELIHOOD-WEIGHTING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x}, w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight

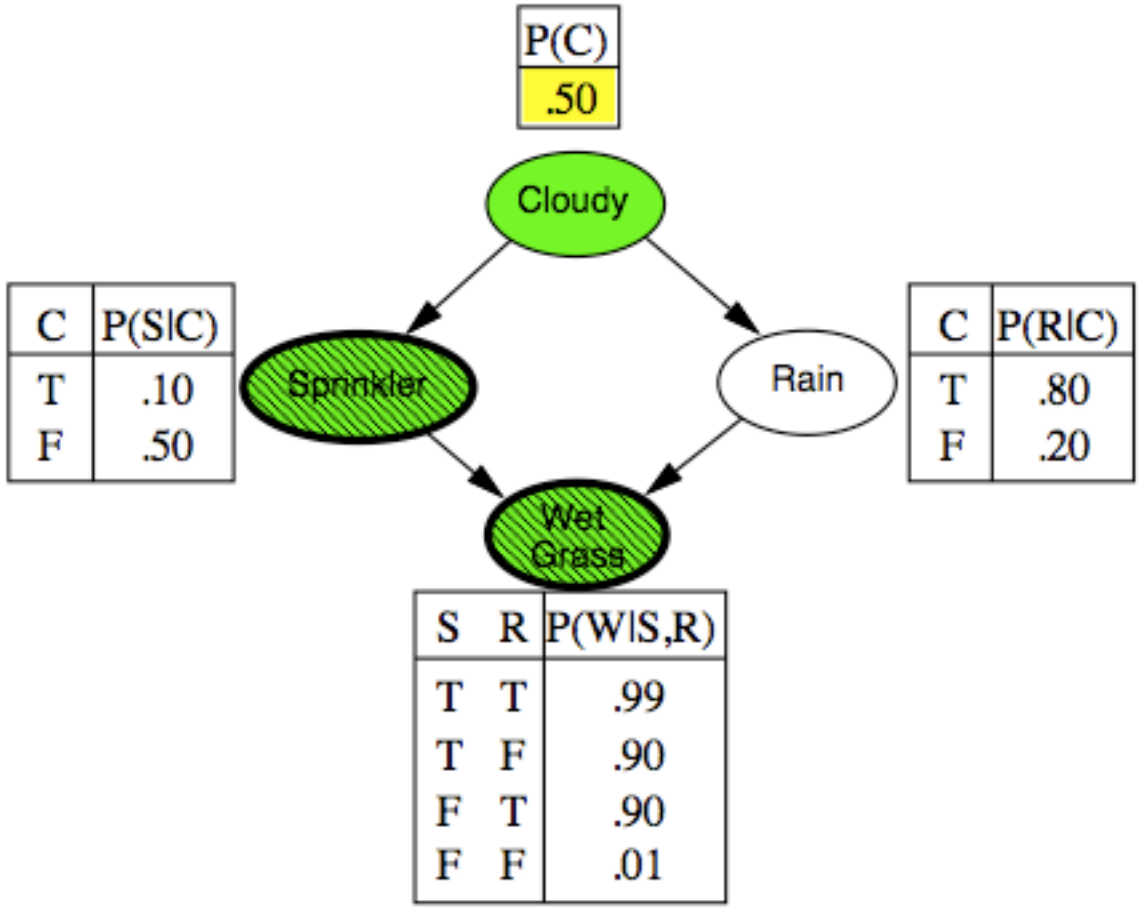
  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for $i = 1$ **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$, w

Example


P(C)
.50
Cloudy
C P(S|C)
T .10
F .50
Sprinkler
Rain
C P(R|C)
T .80
F .20
Wet Grass
S R P(W|S,R)
T T .99
T F .90
F T .90
F F .01

$w = 1.0$

Example


P(C)
.50
Cloudy
C P(S|C)
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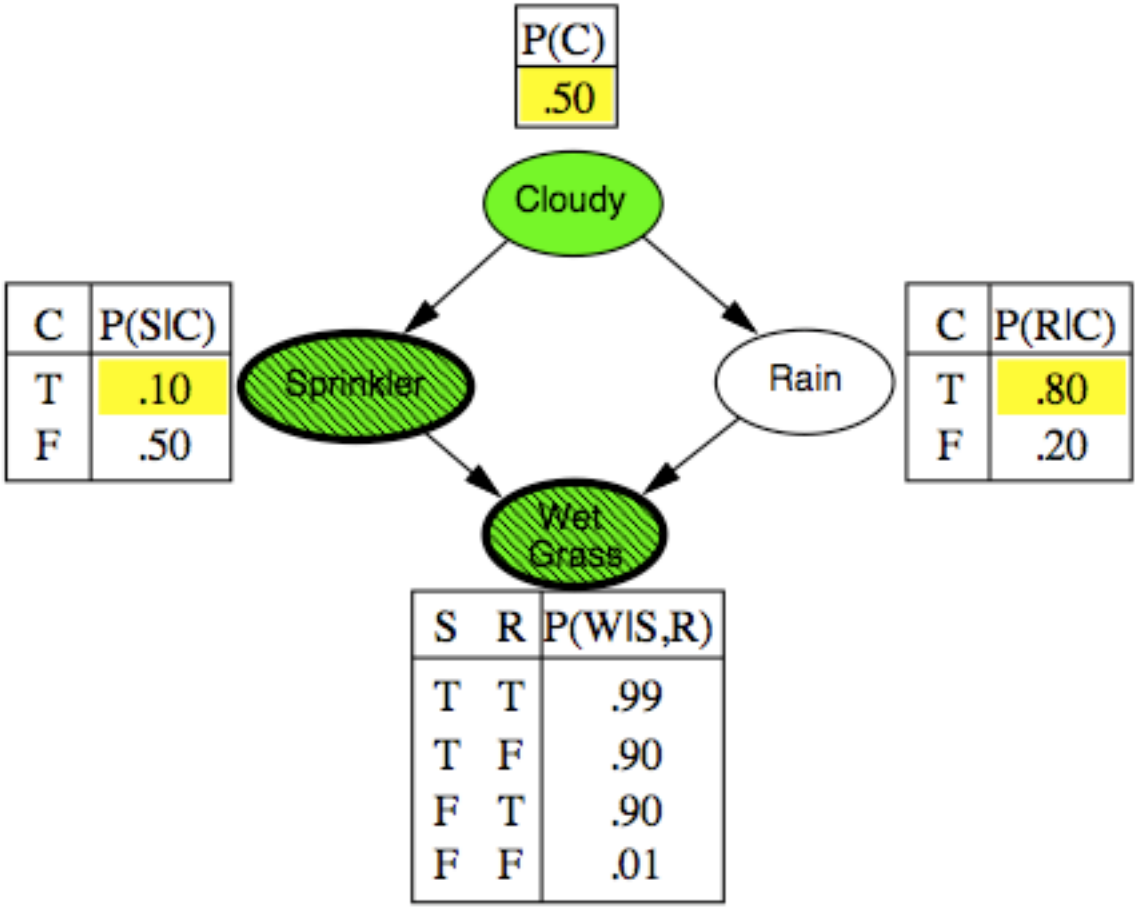
C P(R|C)
T .80
F .20
Wet Grass
S R P(W|S,R)
T T .99
T F .90
F T .90
F F .01

$w = 1.0$

Example


P(C)
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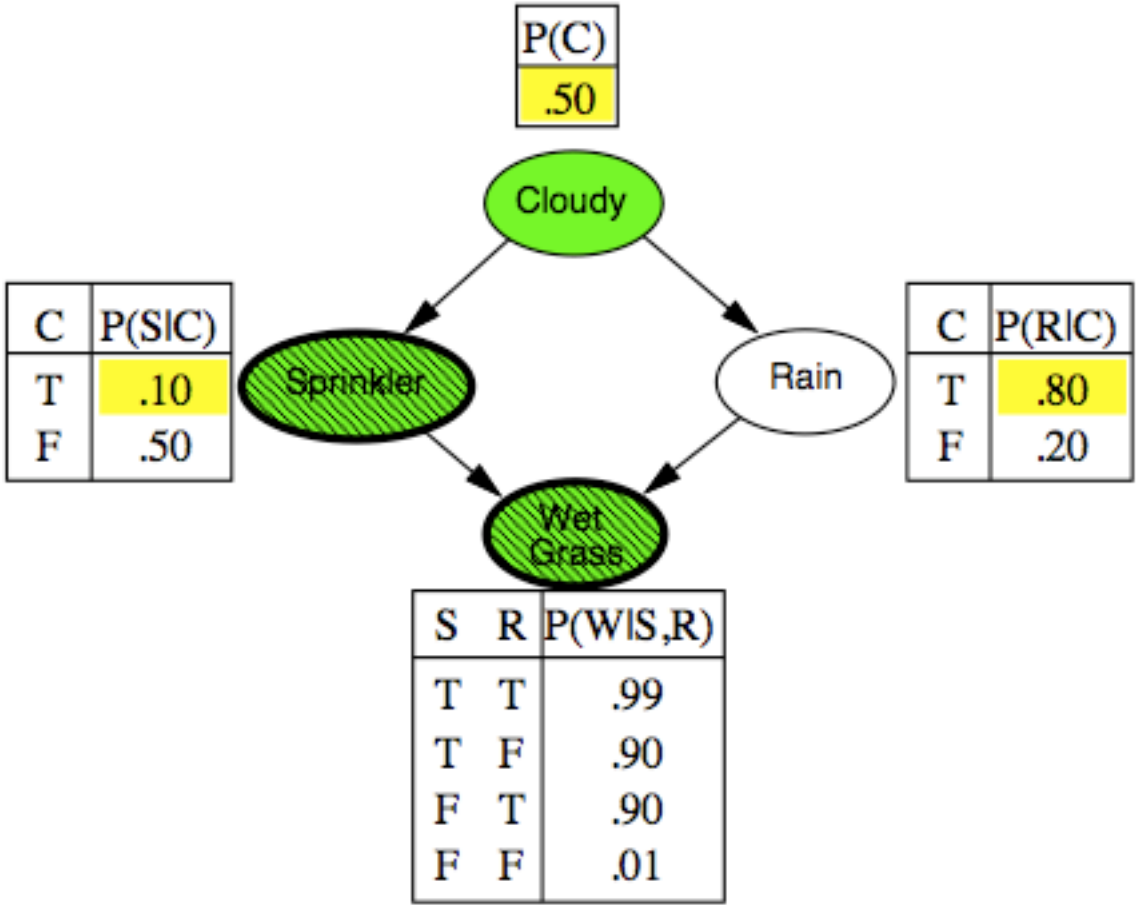
T .80
F .20
Wet Grass
S R P(W|S,R)
T T .99
T F .90
F T .90
F F .01

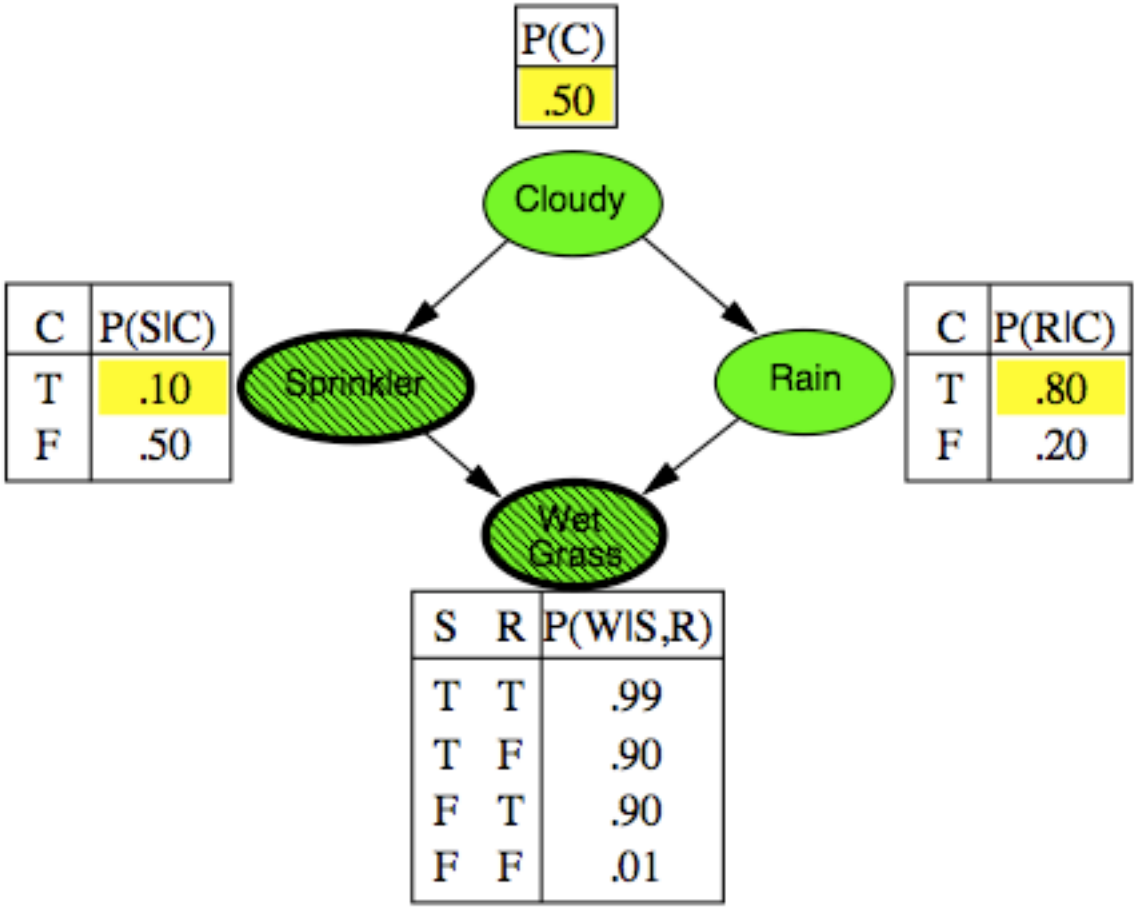
$w = 1.0$

Example



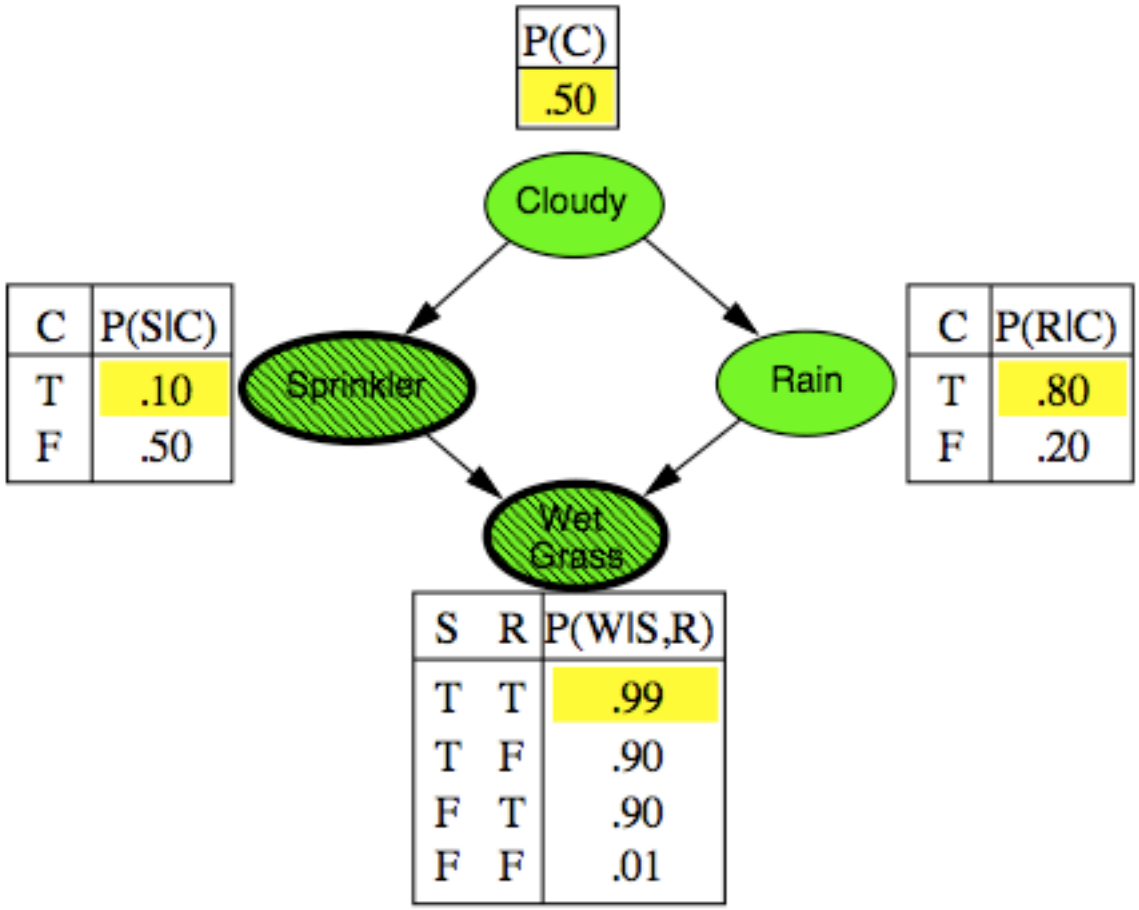
$w = 1.0 \times 0.1$

Example



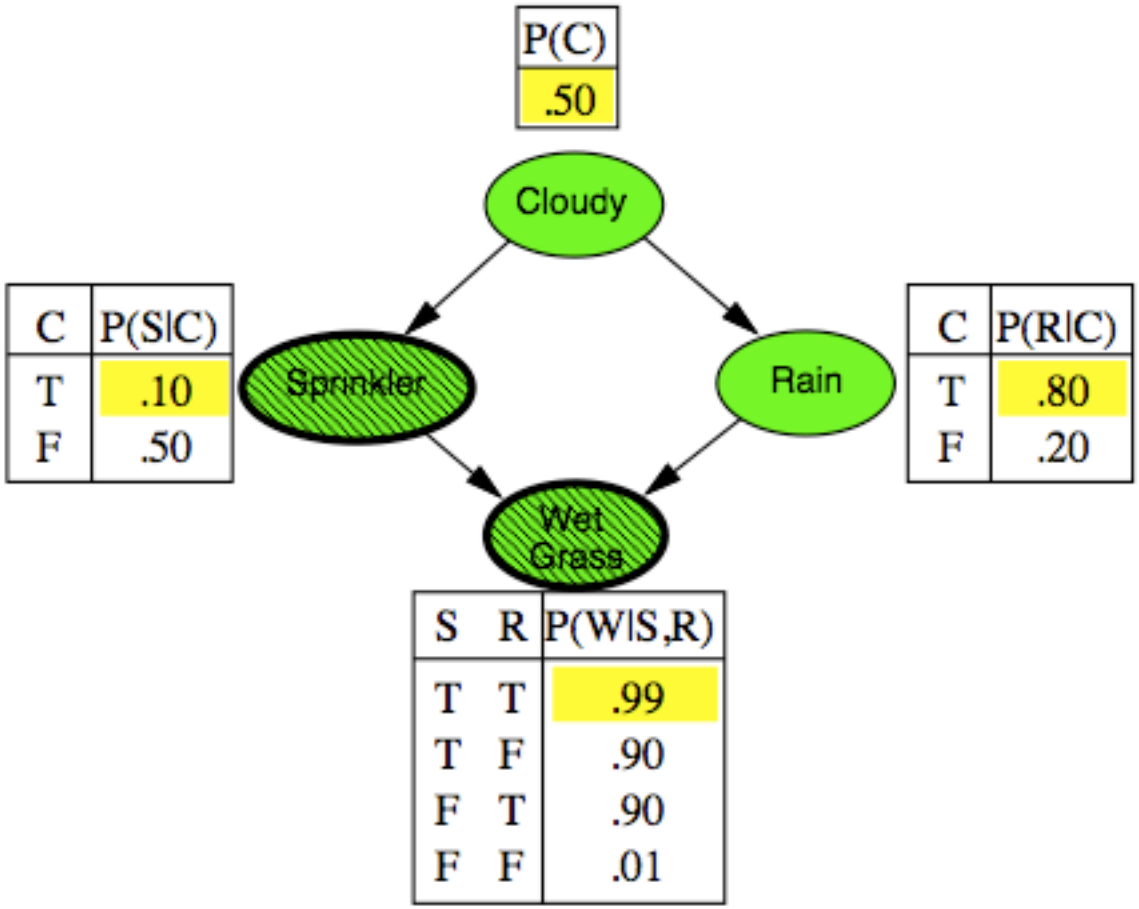
$w = 1.0 \times 0.1$

Example



$w = 1.0 \times 0.1$

Example



$w = 1.0 \times 0.1 \times 0.99 = 0.099$

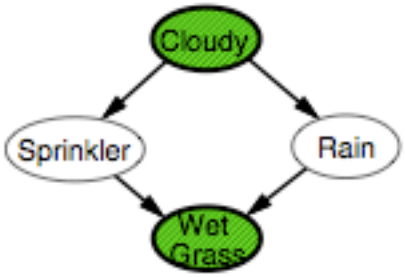
Likelihood weighting analysis

Sampling probability for WEIGHTEDSAMPLE is

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | parents(Z_i))$$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | parents(E_i))$$

Weighted sampling probability is

$$\begin{aligned} & S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e}) \\ & = \prod_{i=1}^{l} P(z_i | parents(Z_i)) \prod_{i=1}^{m} P(e_i | parents(E_i)) \\ & = P(\mathbf{z}, \mathbf{e}) \text{ (by standard global semantics of network)} \end{aligned}$$

Hence likelihood weighting returns consistent estimates but performance still degrades with many evidence variables because a few samples have nearly all the total weight

Approximate Inference using MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

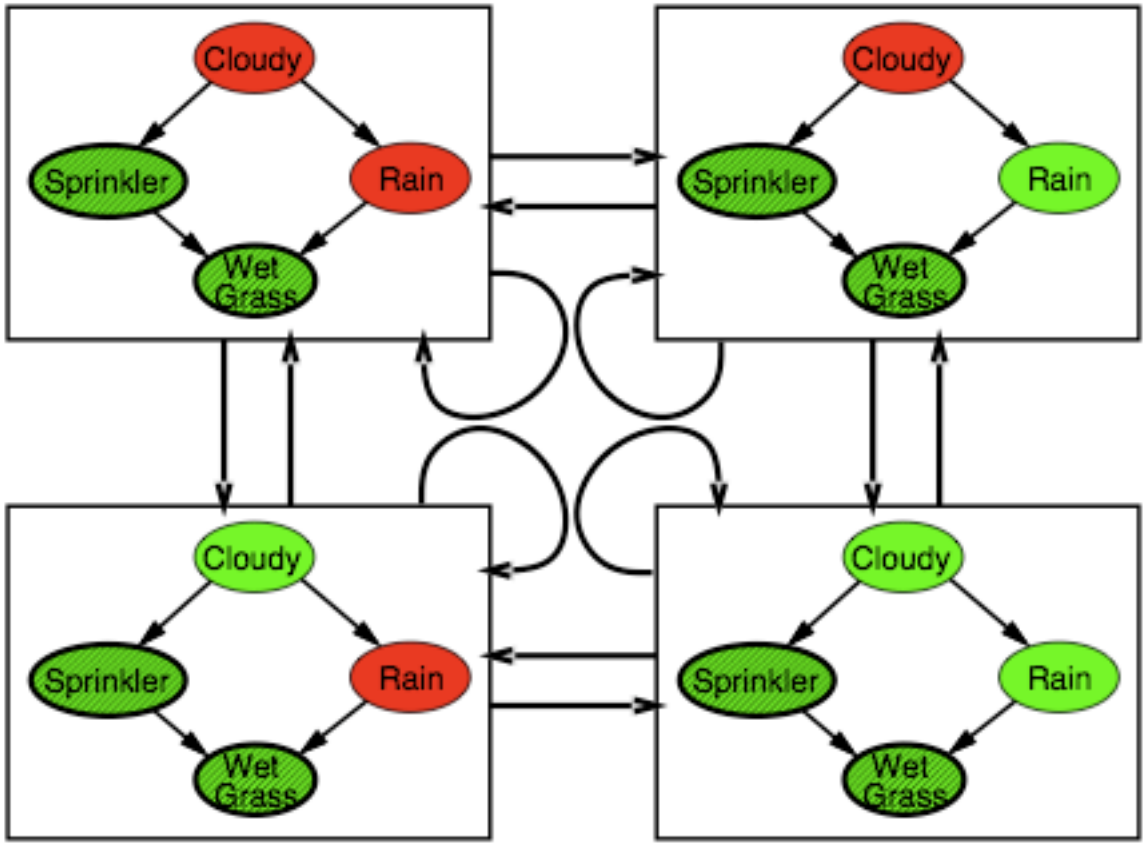
function MCMC-ASK($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

  initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
  for $j = 1$ to N **do**
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{x}$ from $\mathbf{P}(Z_i|mb(Z_i))$
        given the values of $MB(Z_i)$ in $\mathbf{x}$
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

Can also choose a variable to sample at random each time

Markov Chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC continued

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

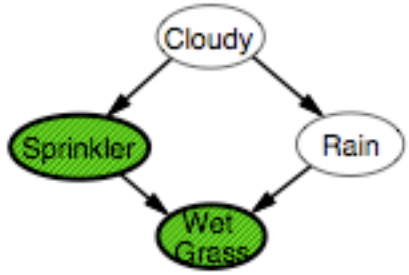
E.g., visit 100 states
31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
$= \textsc{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$

Theorem: chain approaches stationary distribution:
long-run fraction of time spent in each state is exactly proportional to its posterior probability

Markov Blanket Sampling

Markov blanket of $Cloudy$ is
$Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
$Cloudy$, $Sprinkler$, and $WetGrass$



Probability given the Markov blanket is calculated as follows:

$$P(x_i'|mb(X_i)) = P(x_i'|parents(X_i))\Pi_{Z_j \in Children(X_i)} P(z_j|parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
   $P(X_i|mb(X_i))$ won't change much (law of large numbers)

Summary

Exact inference by variable elimination:

- polytime on polytrees, NP-hard on general graphs
- space = time, very sensitive to topology

Approximate inference by LW, MCMC:

- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables

- Direct Sampling
- Rejection Sampling
- Importance Sampling
- MCMC

- Additional inference techniques
- Loopy belief propagation
- Variational inference