

## Simply-Typed Lambda Calculus with Extensions

$t ::= x \mid \lambda x:T.t \mid (t \ t) \mid \text{true} \mid \text{false} \mid \text{if } t \ t \ t \mid \sim t \mid t = t$   
 $\mid \langle \# \rangle \mid t + t \mid t - t \mid t^* t \mid t < t \mid t > t$   
 $\mid \text{nil } T \mid \text{cons } t \ t \mid \text{isnil } t \mid \text{head } t \mid \text{tail } t$   
 $\mid \text{fix } t$

$T ::= T \rightarrow T \mid \mathbb{Z} \mid \mathbb{B} \mid \llbracket T \rrbracket$

$v ::= \lambda x:T.t \mid \text{true} \mid \text{false} \mid \langle \# \rangle \mid \text{nil } T \mid \text{cons } t \ t$

### Evaluation Rules

**E-App1**  $\frac{t_1 \rightarrow t'_1}{(t_1 \ t_2) \rightarrow (t'_1 \ t_2)}$

**E-App2**  $\frac{t_2 \rightarrow t'_2}{(v \ t_2) \rightarrow (v \ t'_2)}$

**E-App-Abs**  $\frac{}{((\lambda x:T.t) \ v) \rightarrow (t[x \mapsto v])}$

**E-If**  $\frac{t_1 \rightarrow t'_1}{\text{if } t_1 \ t_2 \ t_3 \rightarrow \text{if } t'_1 \ t_2 \ t_3}$

**E-If-true**  $\frac{}{\text{if true } t_2 \ t_3 \rightarrow t_2}$

**E-If-false**  $\frac{}{\text{if false } t_2 \ t_3 \rightarrow t_3}$

**E-Neg1**  $\frac{t \rightarrow t'}{\sim t \rightarrow \sim t'}$

**E-Neg-T**  $\frac{}{\sim \text{true} \rightarrow \text{false}}$

**E-Neg-F**  $\frac{}{\sim \text{false} \rightarrow \text{true}}$

**E-Eq1**  $\frac{t_1 \rightarrow t'_1}{t_1 = t_2 \rightarrow t'_1 = t_2}$

**E-Eq2**  $\frac{t_2 \rightarrow t'_2}{v = t_2 \rightarrow v = t'_2}$

**E-Eq**  $\frac{\text{ans: are } v_1, v_2 \text{ } \alpha\text{-equivalent?}}{v_1 = v_2 \rightarrow \text{ans}}$

**E-Add1**  $\frac{t_1 \rightarrow t'_1}{t_1 + t_2 \rightarrow t'_1 + t_2}$

**E-Add2**  $\frac{t_2 \rightarrow t'_2}{v + t_2 \rightarrow v + t'_2}$

**E-Add**  $\frac{}{v_1 + v_2 \rightarrow (\text{perform addition})}$

**E-GT1**  $\frac{t_1 \rightarrow t'_1}{t_1 > t_2 \rightarrow t'_1 > t_2}$

**E-GT2**  $\frac{t_2 \rightarrow t'_2}{v > t_2 \rightarrow v > t'_2}$

**E-GT**  $\frac{}{v_1 > v_2 \rightarrow (\text{perform relation check})}$

*E-Mul/1/2, E-Sub/1/2, E-LT/1/2 follow the same patterns as E-Add/1/2, E-GT/1/2.*

**E-isnil1**  $\frac{t \rightarrow t'}{\text{isnil } t \rightarrow \text{isnil } t'}$

**E-isnil-T**  $\frac{}{\text{isnil } (\text{nil } T) \rightarrow \text{true}}$

**E-isnil-F**  $\frac{}{\text{isnil } (\text{cons } t_1 \ t_2) \rightarrow \text{false}}$

**E-head1**  $\frac{t \rightarrow t'}{\text{head } t \rightarrow \text{head } t'}$

**E-head**  $\frac{}{\text{head } (\text{cons } t_1 \ t_2) \rightarrow t_1}$

**E-tail1**  $\frac{t \rightarrow t'}{\text{tail } t \rightarrow \text{tail } t'}$

**E-tail**  $\frac{}{\text{tail } (\text{cons } t_1 \ t_2) \rightarrow t_2}$

**E-Fix**  $\frac{}{\text{fix } (\lambda x:T.t) \rightarrow t[x \mapsto \text{fix } (\lambda x:T.t)]}$

### Typing Rules

**Ty-Var**  $\frac{(v,T) \in \Gamma}{\Gamma \vdash v:T}$

**Ty- $\lambda$**   $\frac{\Gamma, (x, T_d) \vdash t: T_r}{\Gamma \vdash (\lambda x: T_d. t) : T_d \rightarrow T_r}$

**Ty-app**  $\frac{\Gamma \vdash t_1: T_d \rightarrow T_r \quad \Gamma \vdash t_2: T_d}{\Gamma \vdash (t_1 \ t_2) : T_r}$

**Ty-true**  $\frac{}{\Gamma \vdash \text{true}: \mathbb{B}}$

**Ty-false**  $\frac{}{\Gamma \vdash \text{false}: \mathbb{B}}$

**Ty-if**  $\frac{\Gamma \vdash t_1: \mathbb{B} \quad \Gamma \vdash t_2: T \quad \Gamma \vdash t_3: T}{\Gamma \vdash (\text{if } t_1 \ t_2 \ t_3) : T}$

**Ty-neg**  $\frac{\Gamma \vdash t: \mathbb{B}}{\Gamma \vdash (\sim t) : \mathbb{B}}$

**Ty-eq**  $\frac{\Gamma \vdash t_1: T \quad \Gamma \vdash t_2: T}{\Gamma \vdash (t_1 = t_2) : \mathbb{B}}$

**Ty- $\mathbb{Z}$**   $\frac{}{\Gamma \vdash \langle \# \rangle : \mathbb{Z}}$

**Ty-Add**  $\frac{\Gamma \vdash t_1: \mathbb{Z} \quad \Gamma \vdash t_2: \mathbb{Z}}{\Gamma \vdash (t_1 + t_2) : \mathbb{Z}}$

**Ty-LT**  $\frac{\Gamma \vdash t_1: \mathbb{Z} \quad \Gamma \vdash t_2: \mathbb{Z}}{\Gamma \vdash (t_1 < t_2) : \mathbb{B}}$

*Ty-Sub, Ty-Mul, Ty-GT are similar.*

**Ty-nil**  $\frac{}{\Gamma \vdash (\text{nil } \llbracket T \rrbracket) : \llbracket T \rrbracket}$

**Ty-cons**  $\frac{\Gamma \vdash t_1: T \quad \Gamma \vdash t_2: \llbracket T \rrbracket}{\Gamma \vdash (\text{cons } t_1 \ t_2) : \llbracket T \rrbracket}$

**Ty-head**  $\frac{\Gamma \vdash t: \llbracket T \rrbracket}{\Gamma \vdash (\text{head } t) : T}$

**Ty-tail**  $\frac{\Gamma \vdash t: \llbracket T \rrbracket}{\Gamma \vdash (\text{tail } t) : \llbracket T \rrbracket}$

**Ty-isnil**  $\frac{\Gamma \vdash t: \llbracket T \rrbracket}{\Gamma \vdash (\text{isnil } t) : \mathbb{B}}$

**Ty-fix**  $\frac{\Gamma \vdash t: T \rightarrow T}{\Gamma \vdash (\text{fix } t) : T}$