CS 463: Quiz 5: Simply Typed Lambda Calculus
Spring 2024 **CLOSED NOTES**

Name Answer key
G#: _____

On the back is the target language; it matches our homework language.

1. (5pt). Draw a typing proof tree for the following (well-typed) term: `((λx:𝔹.if x 2 3) true)`

```
----------------Ty-Var ----------------Ty-Z    ----------------Ty-Z
{(x,𝔹)} |- x:B          {(x,𝔹)} |- 2:Z          {(x,𝔹)} |- 3:Z
-------------------------------------------------------------------------Ty-if
{(x,𝔹)} |- (if x 2 3) : Z
-------------------------------------------------------------------------Ty-λ     ---------------Ty-true
{} |- (λx:𝔹.if x 2 3) : B->Z                                                    {} |- true : B
-----------------------------------------------------------------------------------------------------Ty-App
{} |- ((λx:𝔹.if x 2 3) true) :  Z
```

2. (2pt). In the given language, encode the `min2` function. It accepts two ints, and returns the smallest.

```
min2 =    λx:Z. λy:Z. if (x<y) x y
```

3. (3pt). In the given language, encode the `all` function. It accepts a list of Booleans, and answers whether all values in the list were `true`. (Thus, all of an empty list is true as there were no falses found).

```
all =     fix (λ self : [B] -> B.
              λ bs:[B] .
                  if (isnil bs)
                     true
                     (if (~ (head bs))
                         false
                         (self (tail bs))  )  )
```

Simply-Typed Lambda Calculus with Extensions

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t ::= x | λx:T.t | (t t) | true | false | if t t t | ~t | t = t
| <#> | t + t | t-t | t*t | t<t | t>t
| nil T | cons t t | isnil t | head t | tail t
| fix t

T ::= T → T | ℤ | 𝔹 | ⟦ T ⟧

v ::= λx:T.t | true | false | <#> | nil T | cons t t

Evaluation Rules

E-App1 $\frac{t_1 \to t_1'}{(t_1\ t_2) \to (t_1'\ t_2)}$ **E-App2** $\frac{t_2 \to t_2'}{(v\ t_2) \to (v\ t_2')}$ **E-App-Abs** $\frac{}{((\lambda x{:}T.t)\ v) \to (t[x \mapsto v])}$

E-If $\frac{t_1 \to t_1'}{if\ t_1\ t_2\ t_3 \to if\ t_1'\ t_2\ t_3}$ **E-If-true** $\frac{}{if\ true\ t_2\ t_3 \to t_2}$ **E-If-false** $\frac{}{if\ false\ t_2\ t_3 \to t_3}$

E-Neg1 $\frac{t \to t'}{\sim t \to \sim t'}$ **E-Neg-T** $\frac{}{\sim true \to false}$ **E-Neg-F** $\frac{}{\sim false \to true}$

E-Eq1 $\frac{t_1 \to t_1'}{t_1 = t_2 \to t_1' = t_2}$ **E-Eq2** $\frac{t_2 \to t_2'}{v = t_2 \to v = t_2'}$ **E-Eq** $\frac{ans{:}\ are\ v_1, v_2\ \alpha{-}equivalent?}{v_1 = v_2 \to ans}$

E-Add1 $\frac{t_1 \to t_1'}{t_1 + t_2 \to t_1' + t_2}$ **E-Add2** $\frac{t_2 \to t_2'}{v + t_2 \to v + t_2'}$ **E-Add** $\frac{}{v_1 + v_2 \to (perform\ addition)}$

E-GT1 $\frac{t_1 \to t_1'}{t_1 > t_2 \to t_1' > t_2}$ **E-GT2** $\frac{t_2 \to t_2'}{v > t_2 \to v > t_2'}$ **E-GT** $\frac{}{v_1 > v_2 \to (perform\ relation\ check)}$

E-Mul/1/2, E-Sub/1/2, E-LT/1/2 follow the same patterns as E-Add/1/2, E-GT/1/2.

E-isnil1 $\frac{t \to t'}{isnil\ t \to isnil\ t'}$ **E-isnil-T** $\frac{}{isnil\ (nil\ T) \to true}$ **E-isnil-F** $\frac{}{isnil\ (cons\ t_1\ t_2) \to false}$

E-head1 $\frac{t \to t'}{head\ t \to head\ t'}$ **E-head** $\frac{}{head\ (cons\ t_1\ t_2) \to t_1}$ **E-tail1** $\frac{t \to t'}{tail\ t \to tail\ t'}$ **E-tail** $\frac{}{tail\ (cons\ t_1\ t_2) \to t_2}$

E-Fix $\frac{}{fix\ (\lambda self{:}T.t) \to t[self \mapsto fix(\lambda self{:}T.t)]}$

Typing Rules

Ty-Var $\frac{(v,T) \in \Gamma}{\Gamma \vdash v{:}T}$ Ty-λ $\frac{\Gamma, (x, T_d) \vdash t{:}T_r}{\Gamma \vdash (\lambda x{:}T_d.t) : T_d \to T_r}$ Ty-app $\frac{\Gamma \vdash t_1{:}T_d \to T_r \quad \Gamma \vdash t_2{:}T_d}{\Gamma \vdash (t_1\ t_2) : T_r}$

Ty-true $\frac{}{\Gamma \vdash true{:}\ \mathbb{B}}$ Ty-false $\frac{}{\Gamma \vdash false{:}\ \mathbb{B}}$ Ty-if $\frac{\Gamma \vdash t_1{:}\mathbb{B} \quad \Gamma \vdash t_2{:}T \quad \Gamma \vdash t_3{:}T}{\Gamma \vdash (if\ t_1\ t_2\ t_3) : T}$ Ty-neg $\frac{\Gamma \vdash t{:}\mathbb{B}}{\Gamma \vdash (\sim t) : \mathbb{B}}$

Ty-eq $\frac{\Gamma \vdash t_1{:}T_{\square} \quad \Gamma \vdash t_2{:}T_{\square}}{\Gamma \vdash (t_1 = t_2) : \mathbb{B}}$

Ty- ℤ $\frac{}{\Gamma \vdash <\#> :\ \mathbb{Z}}$ Ty-Add $\frac{\Gamma \vdash t_1{:}\mathbb{Z} \quad \Gamma \vdash t_2{:}\mathbb{Z}}{\Gamma \vdash (t_1 + t_2) :\ \mathbb{Z}}$ Ty-LT $\frac{\Gamma \vdash t_1{:}\mathbb{Z} \quad \Gamma \vdash t_2{:}\mathbb{Z}}{\Gamma \vdash (t_1 < t_2) :\ \mathbb{B}}$ *Ty-Sub, Ty-Mul, Ty-GT are similar.*

Ty-nil $\frac{}{\Gamma \vdash (nil\ ⟦\ \mathrm{T}\ ⟧) :\ ⟦\ \mathrm{T}\ ⟧}$ Ty-cons $\frac{\Gamma \vdash t_1{:}\mathrm{T} \quad \Gamma \vdash t_2{:}⟦\mathrm{T}⟧}{\Gamma \vdash (cons\ t_1\ t_2) :\ ⟦\mathrm{T}⟧}$ Ty-head $\frac{\Gamma \vdash t{:}⟦\mathrm{T}⟧}{\Gamma \vdash (head\ t) : \mathrm{T}}$ Ty-tail $\frac{\Gamma \vdash t{:}⟦\mathrm{T}⟧}{\Gamma \vdash (tail\ t) : ⟦\mathrm{T}⟧}$

Ty-isnil $\frac{\Gamma \vdash t : ⟦\mathrm{T}⟧}{\Gamma \vdash (isnil\ t) :\ \mathbb{B}}$ Ty-fix $\frac{\Gamma \vdash t : T \to T}{\Gamma \vdash (fix\ t) :\ \mathrm{T}}$