CS 483 - Data Structures and Algorithm Analysis

Lecture IV: Chapter 4

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February 15, 2006

Outline

A General Plan for Divide-And-Conquer

1. Decompose a problem instance
2. Solve component problem instances
3. Combine components into composite solution

Example: $\sum_{i=0}^{n} a_i$

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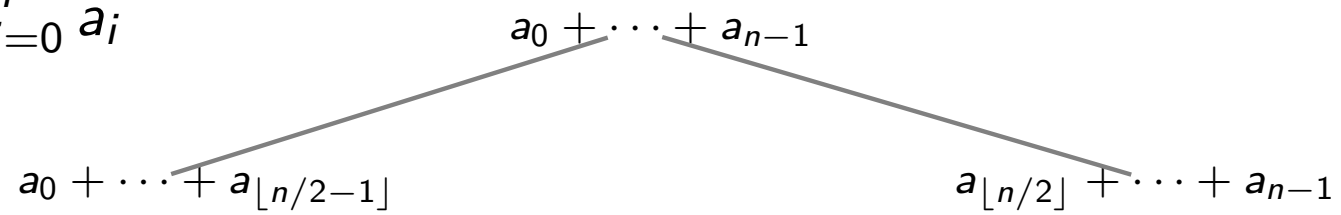

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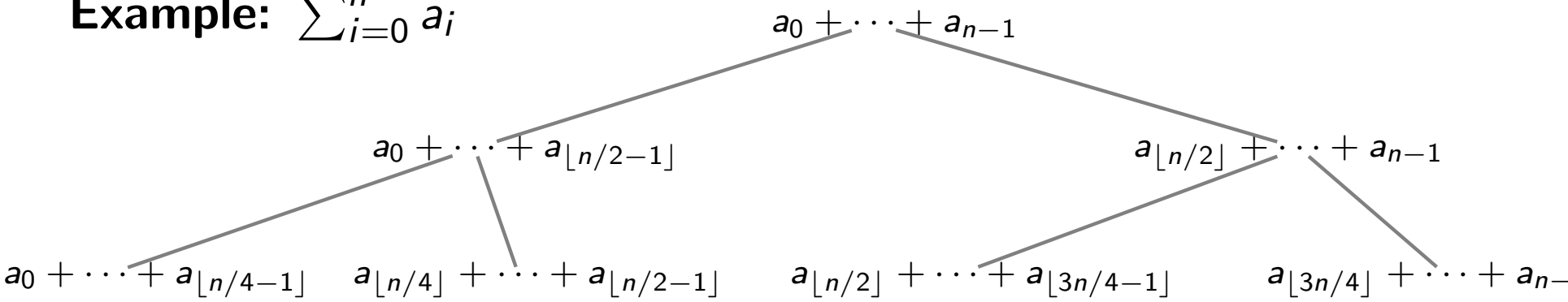

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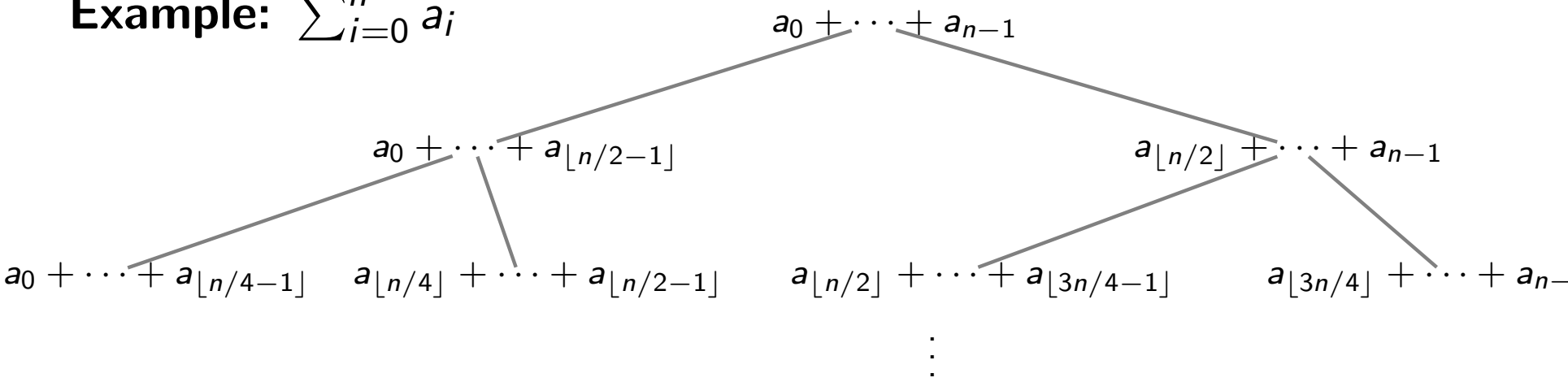

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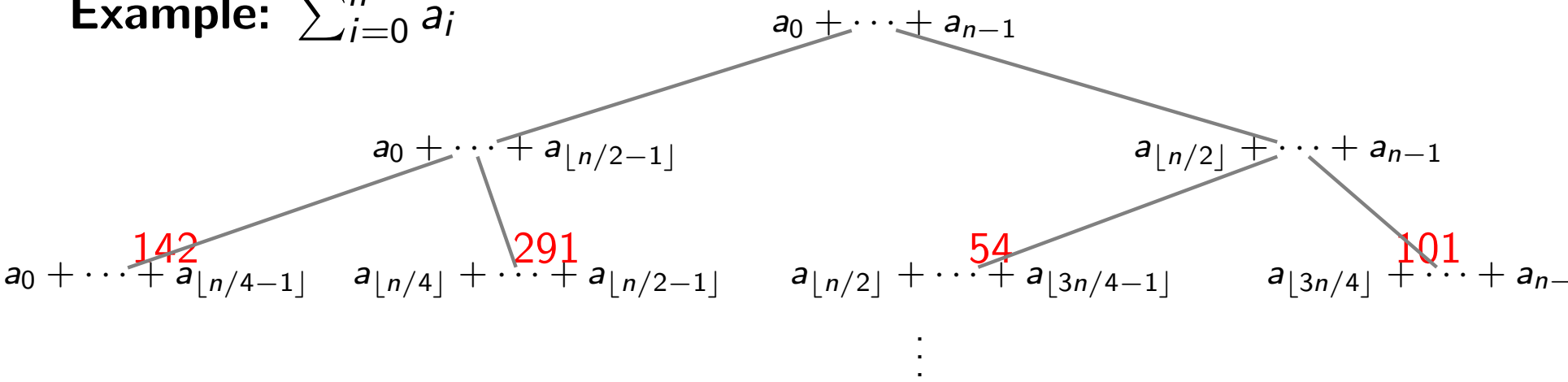

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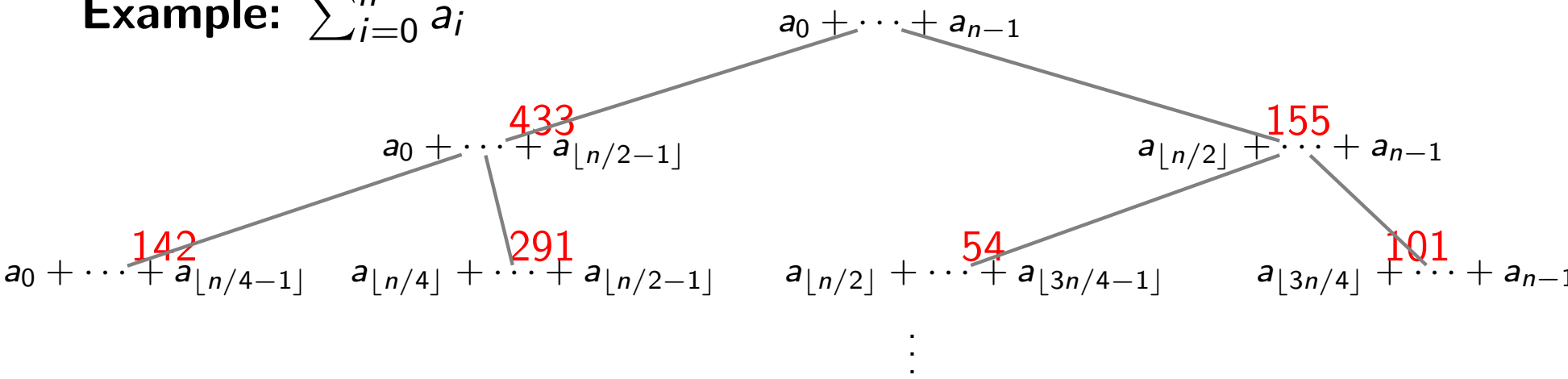

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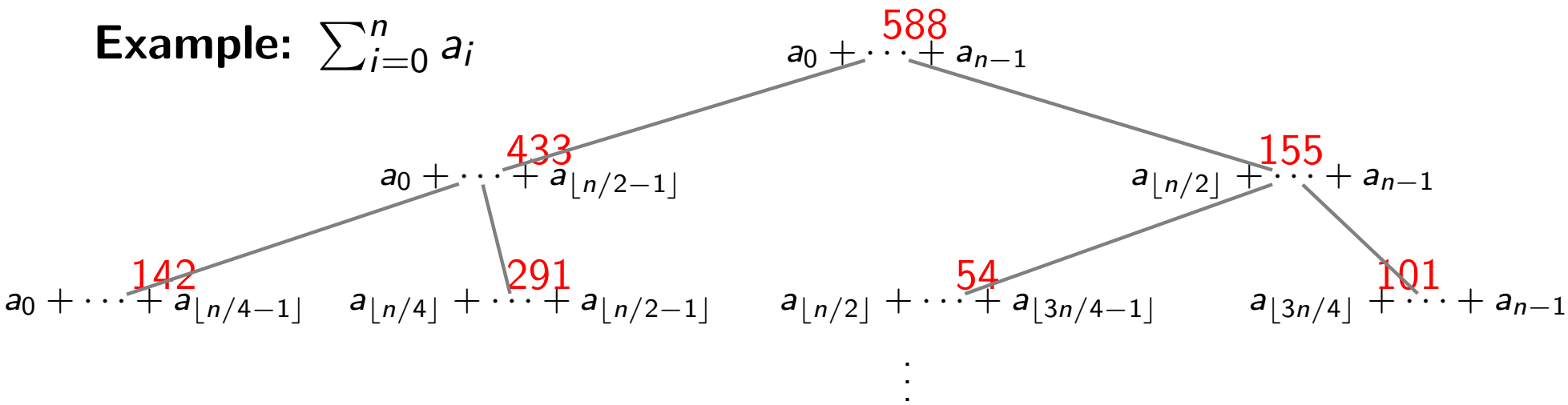

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- Is this D&C example more efficient than brute force?

  No ... it is $\Theta(n)$
- Divide-and-conquer is not necessarily superior ...
- But many times it is, and many of the most efficient algorithms in CS are D&C
- D&C typically involves recursion (at least conceptually)
- D&C is well-suited for parallelization

The Master Theorem

- More generally, a problem of size n can be partitioned into a instances of non-overlapping components of size $\frac{n}{b}$ such that $a \geq 1$ and $b > 1$ (* we assume n is a power of b for simplicity)
- Given this, the *general divide-and-conquer recurrence* can be defined as: $T(n) := aT\left(\frac{n}{b}\right) + f(n)$
- This generalization allows us an analysis short-cut:

Master Theorem

If $f(n) \in \Theta(n^d)$ where $d \geq 0$ in the general divide-and-conquer recurrence then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \lg n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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- Since $a > b^d$, $A(n) \in \Theta(n^{\lg 2}) = \Theta(n)$

Specifying MergeSort

MergeSort($A[0 \ldots n-1]$)

```
if n > 1
   B[0 ... ⌊n/2⌋ − 1] ⟵c A[0 ... ⌊n/2⌋ − 1]
   C[0 ... ⌈n/2⌉ − 1] ⟵c A[⌊n/2⌋ ... n − 1]
   MergeSort(B)
   MergeSort(C)
   Merge(B,C,A)
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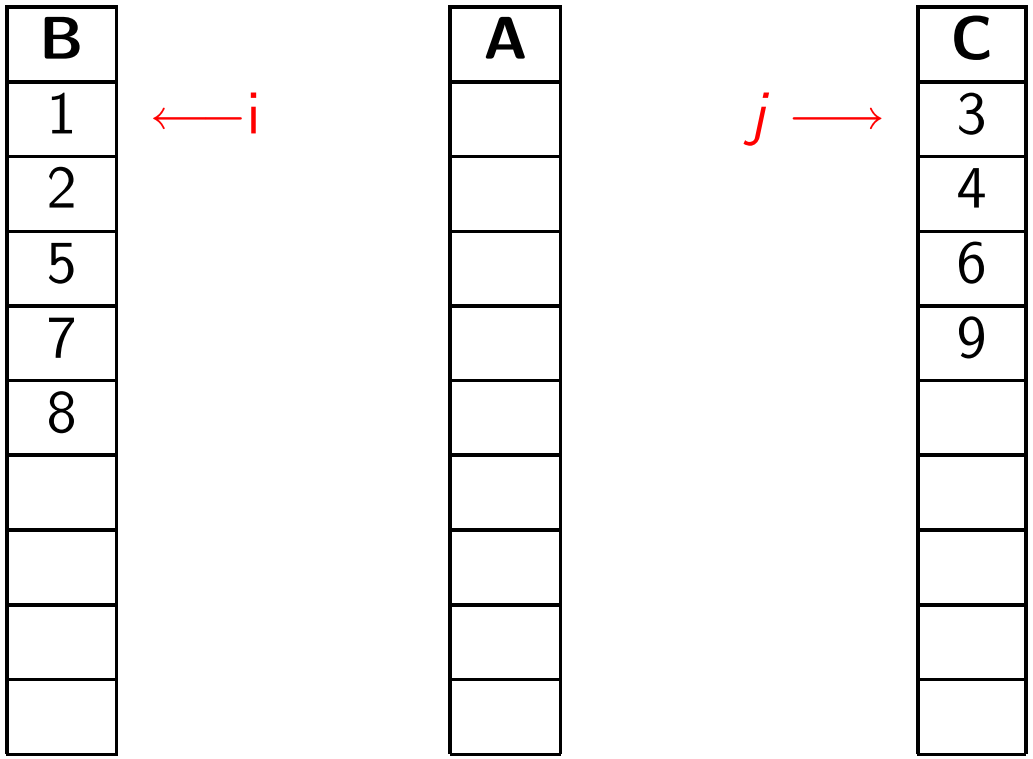
Merge($B[0 \ldots p-1], C[0 \ldots q-1], A[0 \ldots p+q-1]$)

```
i, j, k ⟵ 0
while i < p and j < q do
  if B[i] ≤ C[j] A[k] ⟵ B[i]; i++
  else A[k] ⟵ C[j]; j++
  k++
if i = p A[k ... p + q − 1] ⟵_c C[j ... q − 1]
else A[k ... p + q − 1] ⟵_c B[i ... p − 1]
```

Merging Two Sorted Lists



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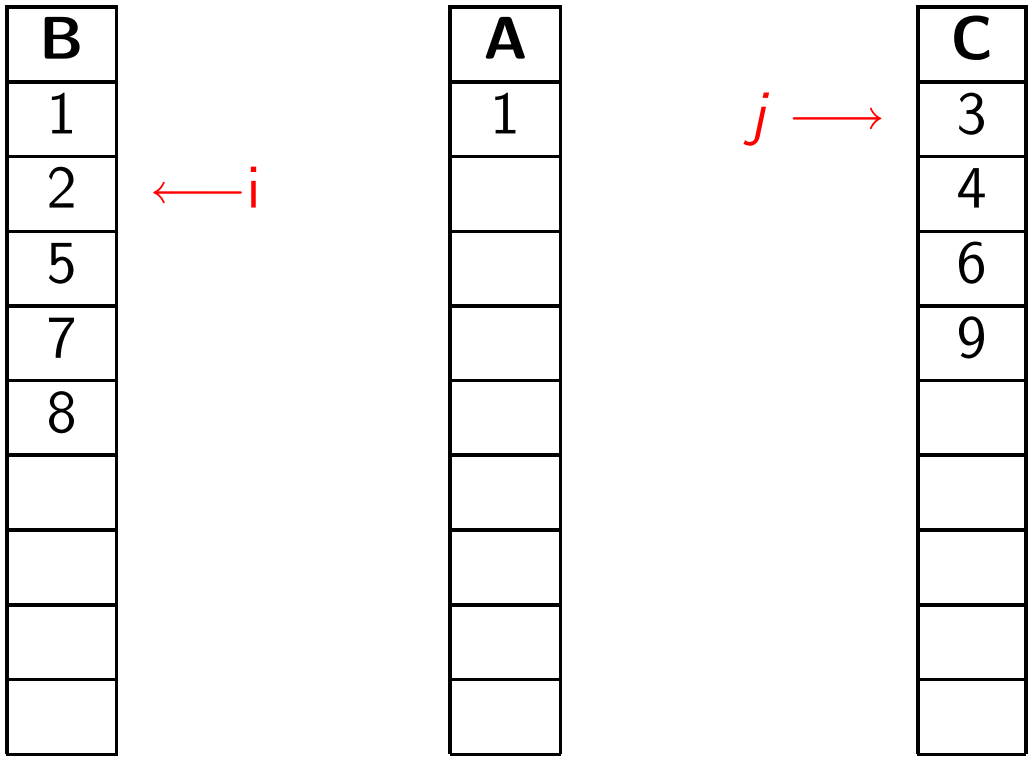

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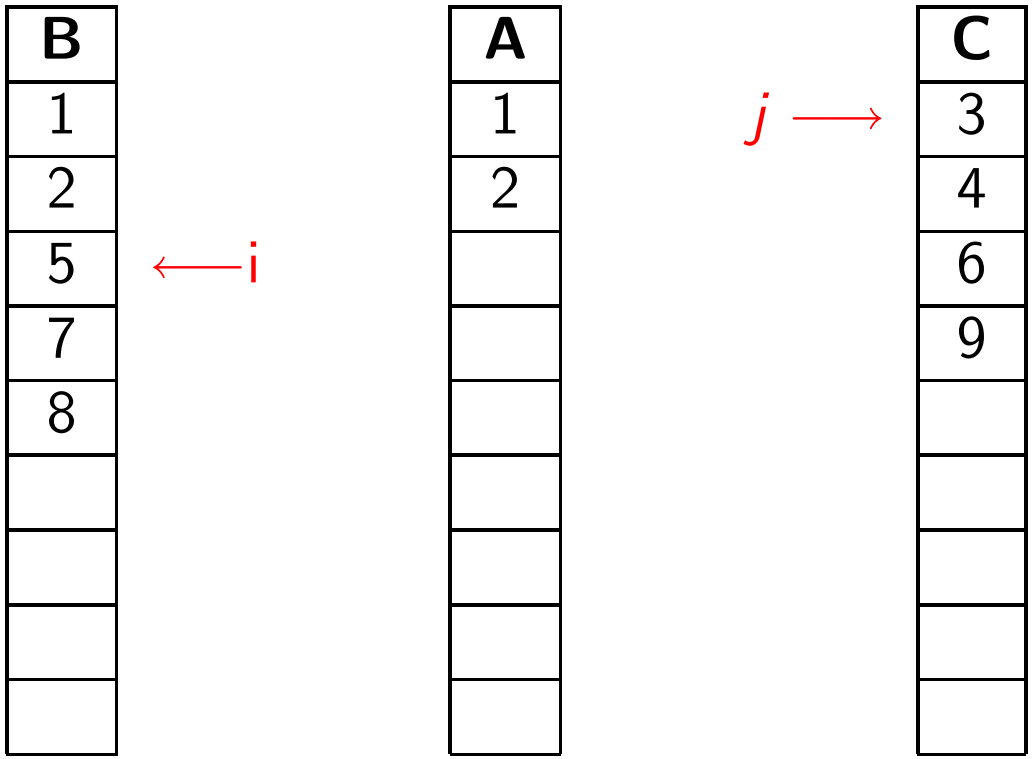

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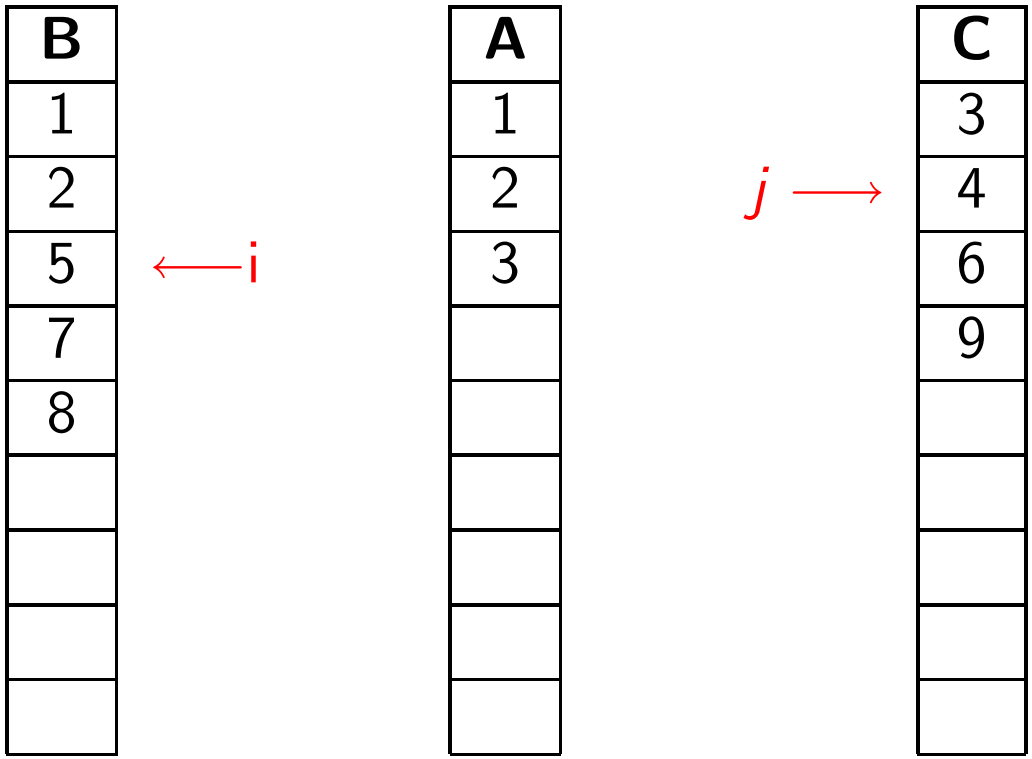

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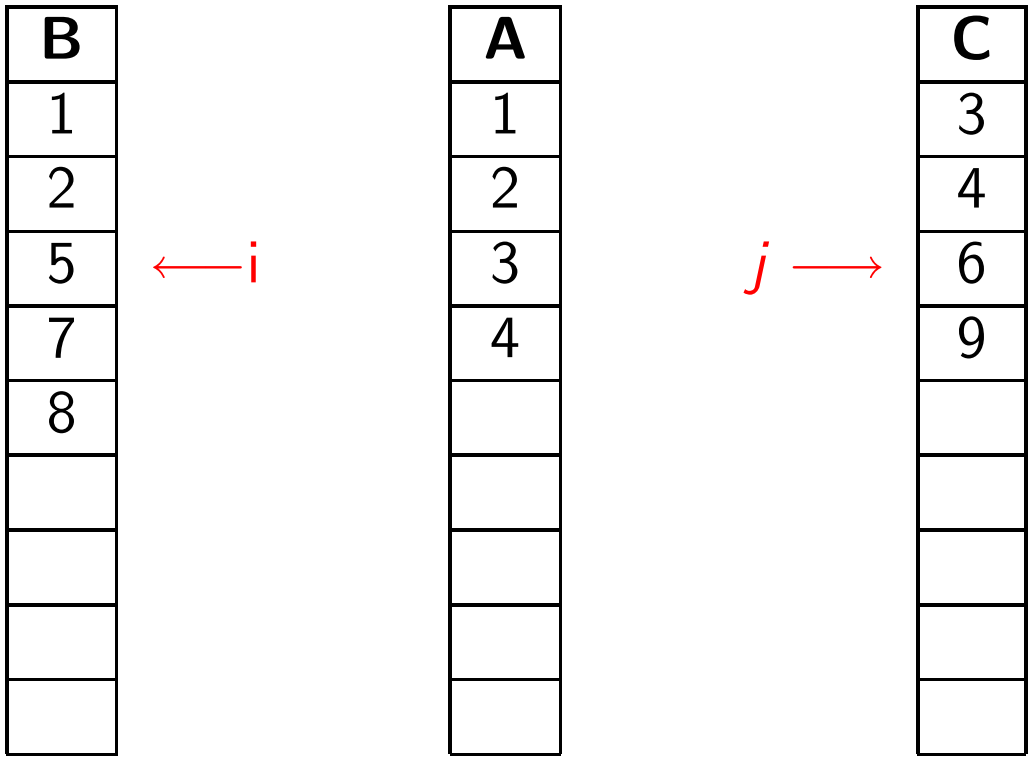

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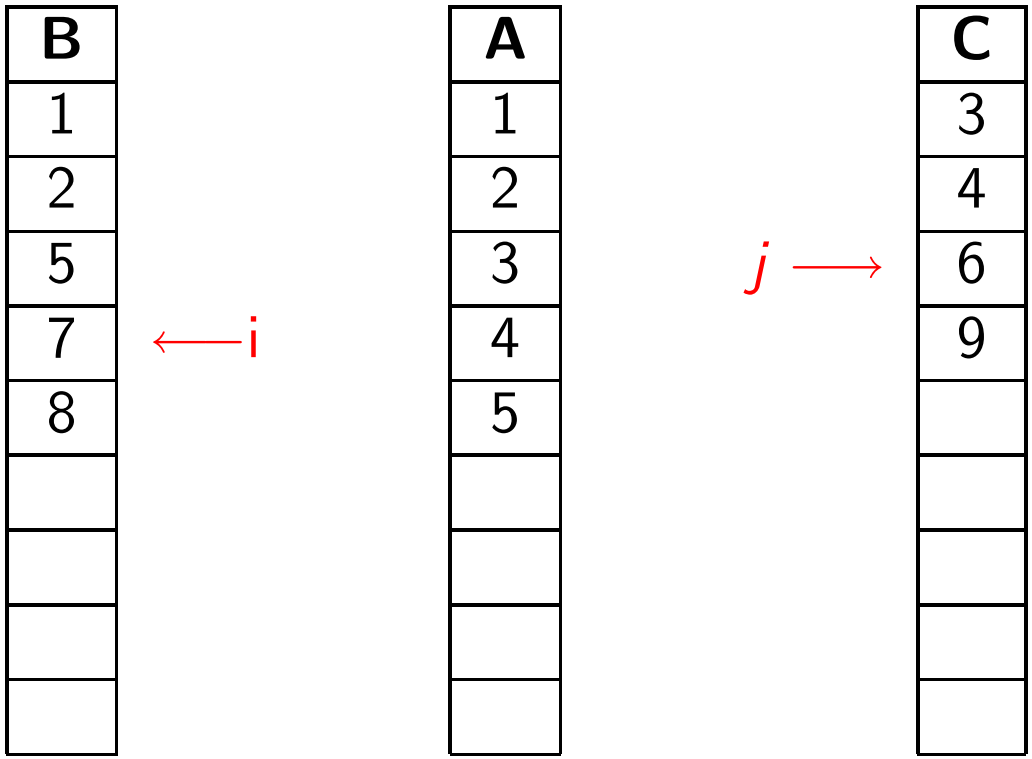

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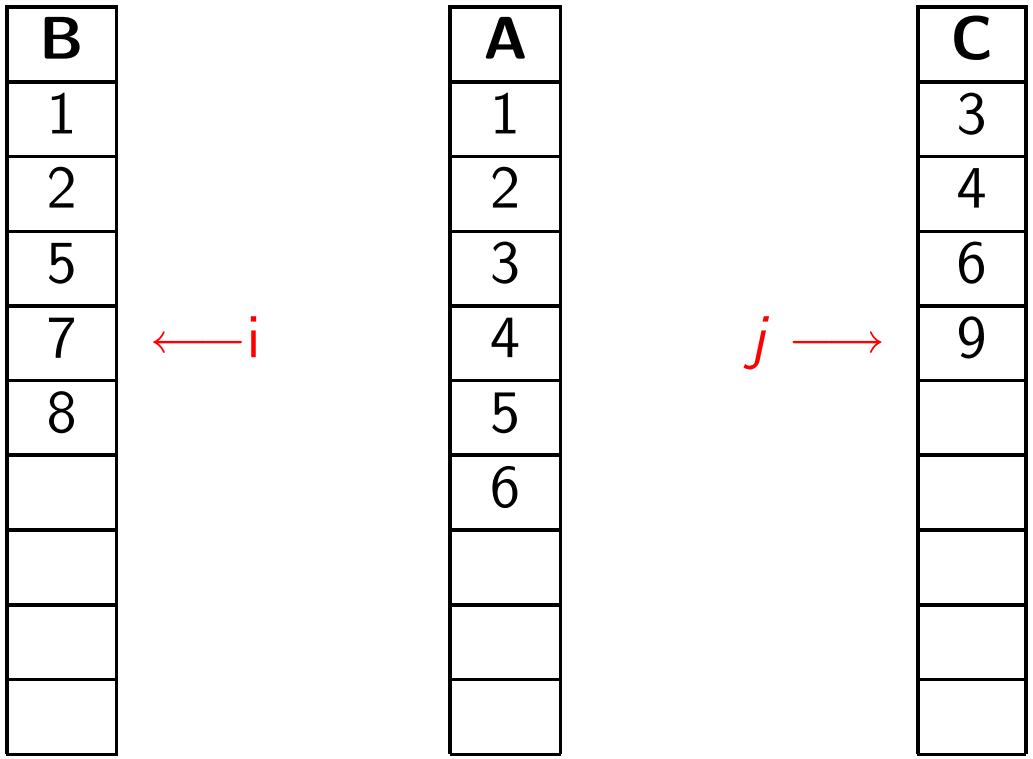

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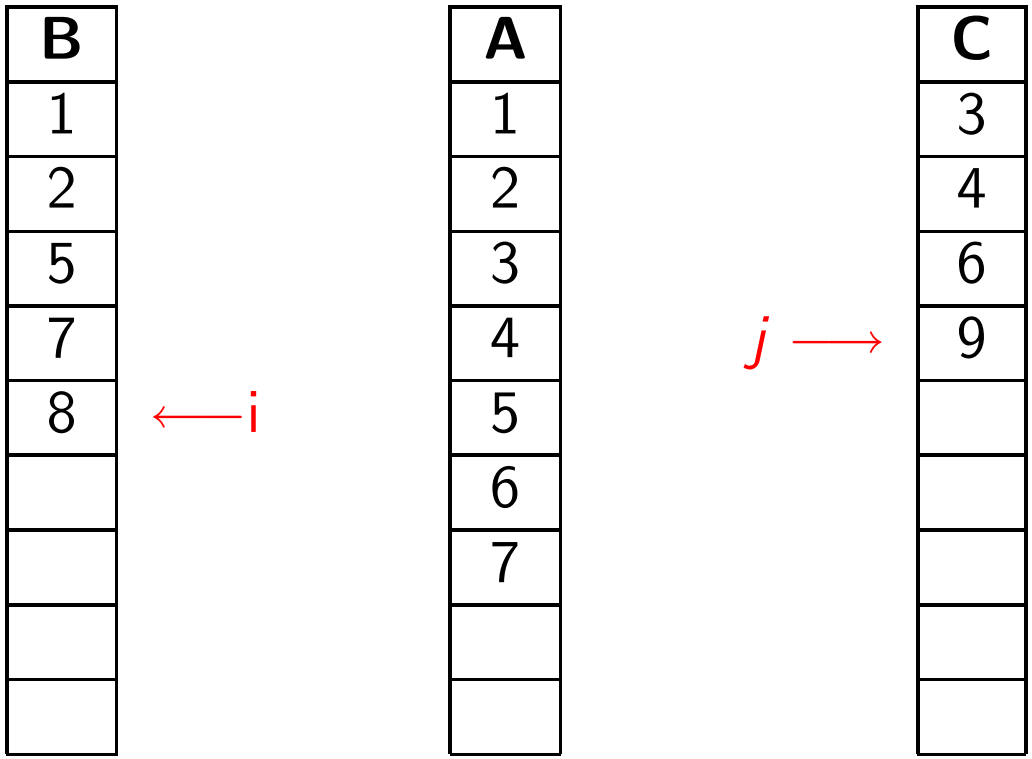

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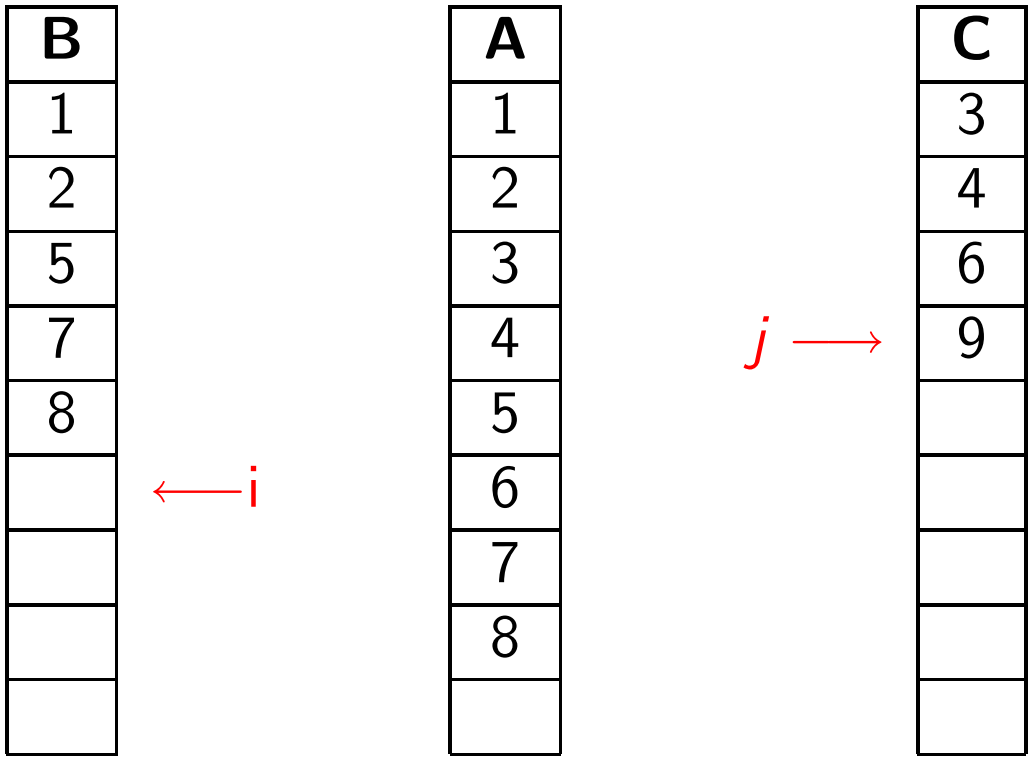

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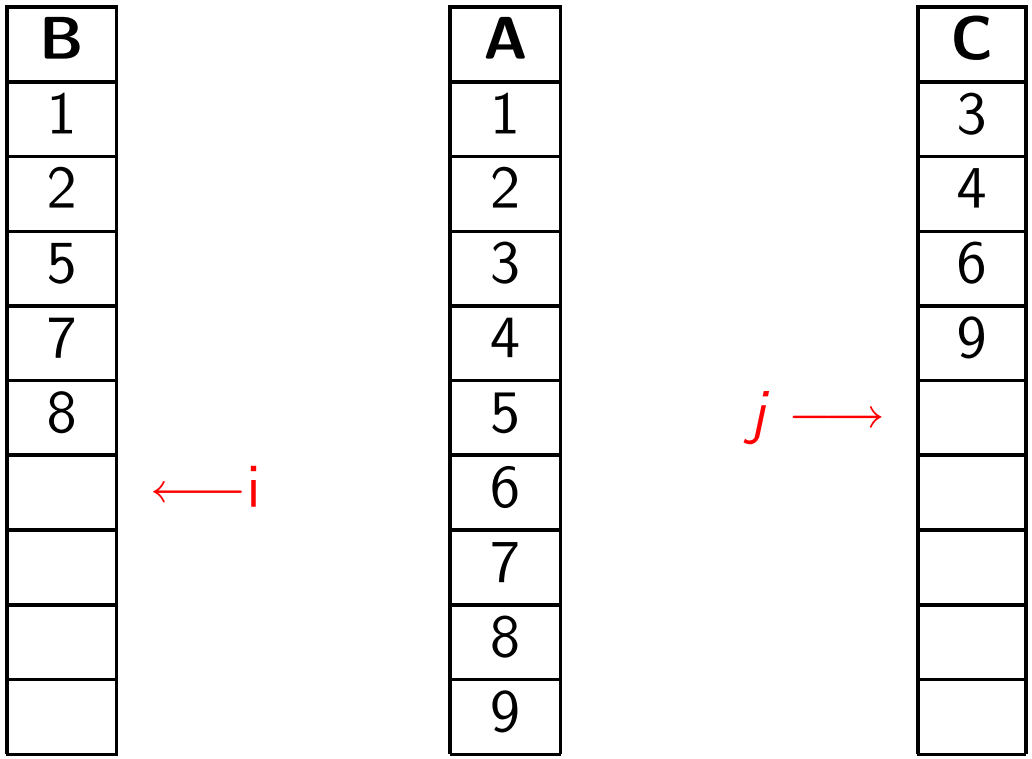

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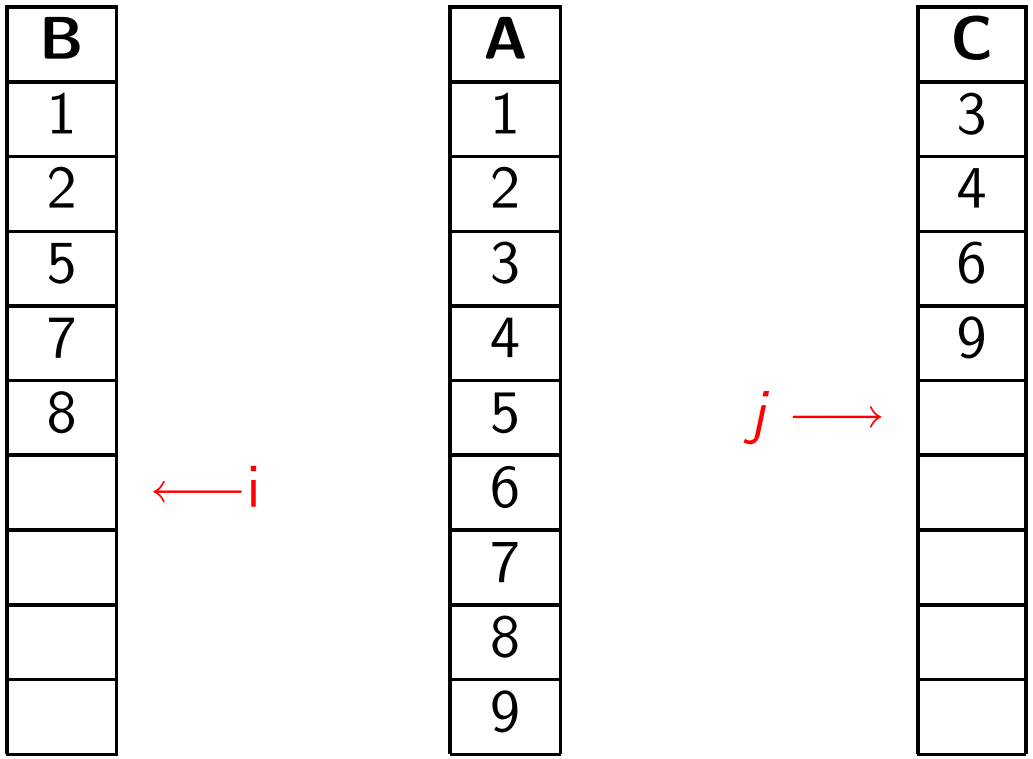

How many comparisons?

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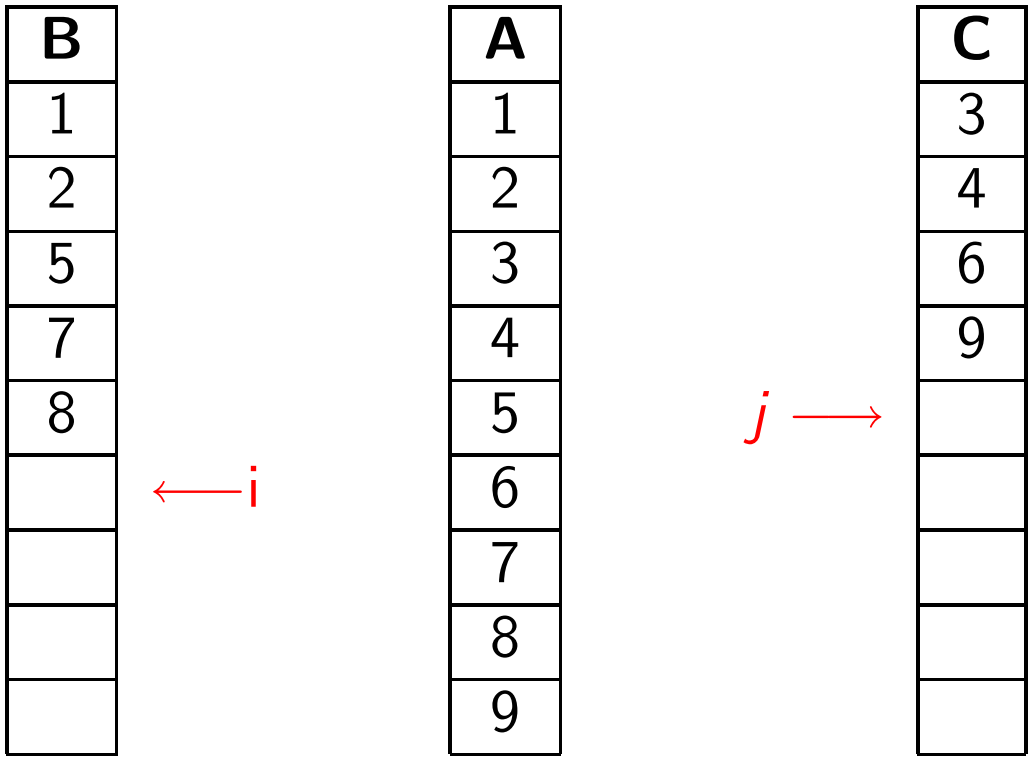

How many comparisons?

$n - 1$

Decomposing & Combining

Decomposing...

8 7 5 1 6 3 2 4

Decomposing & Combining

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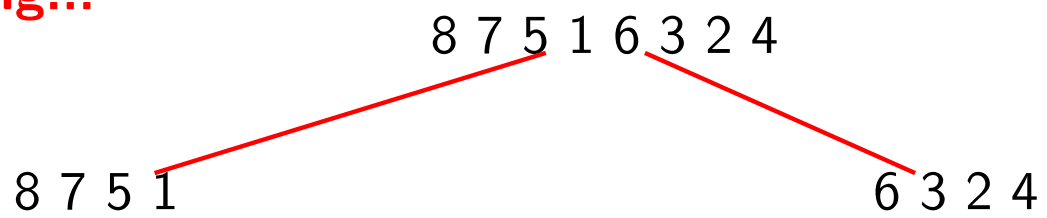

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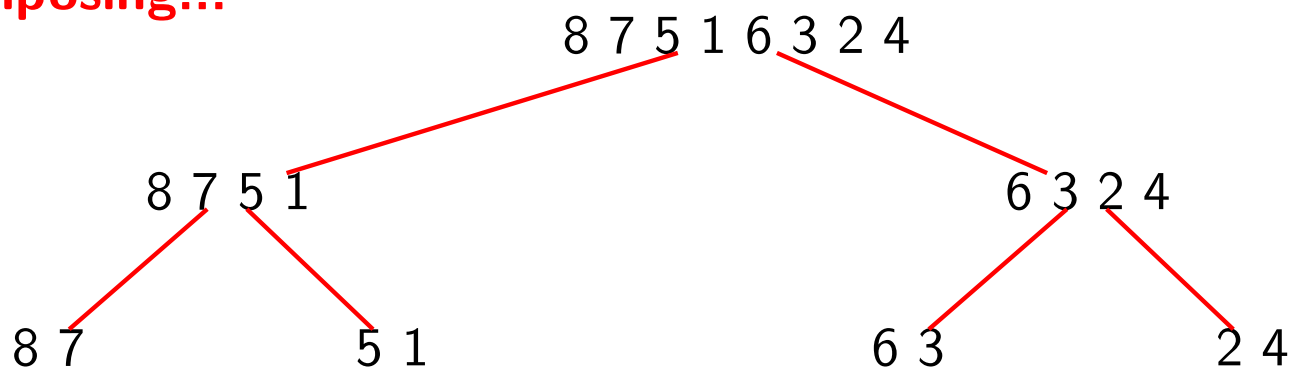

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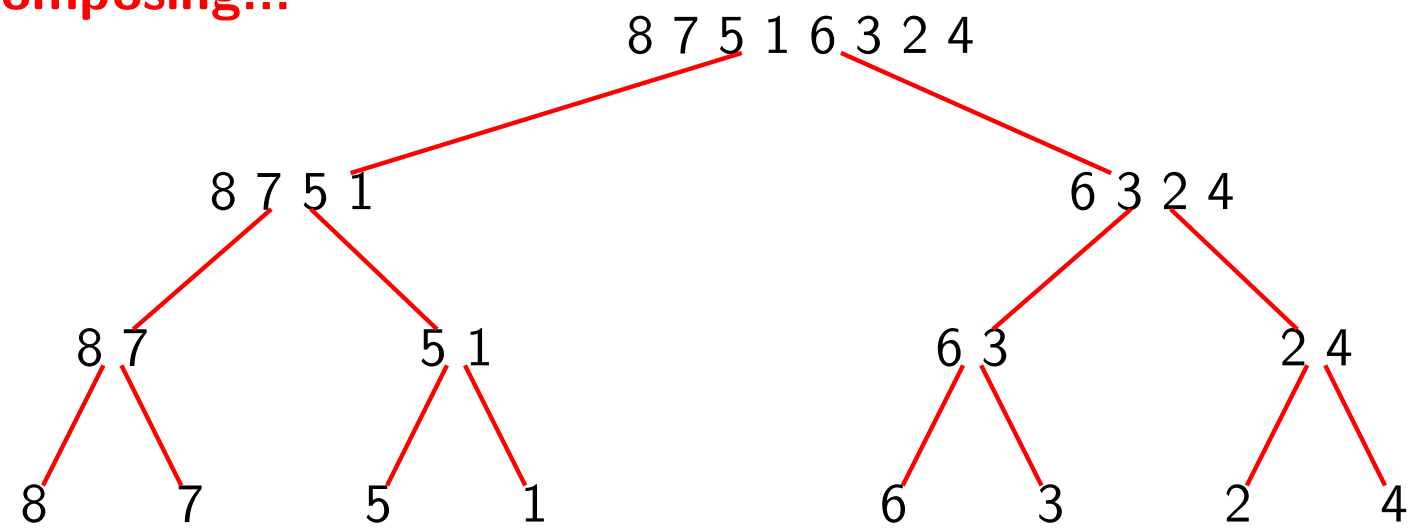

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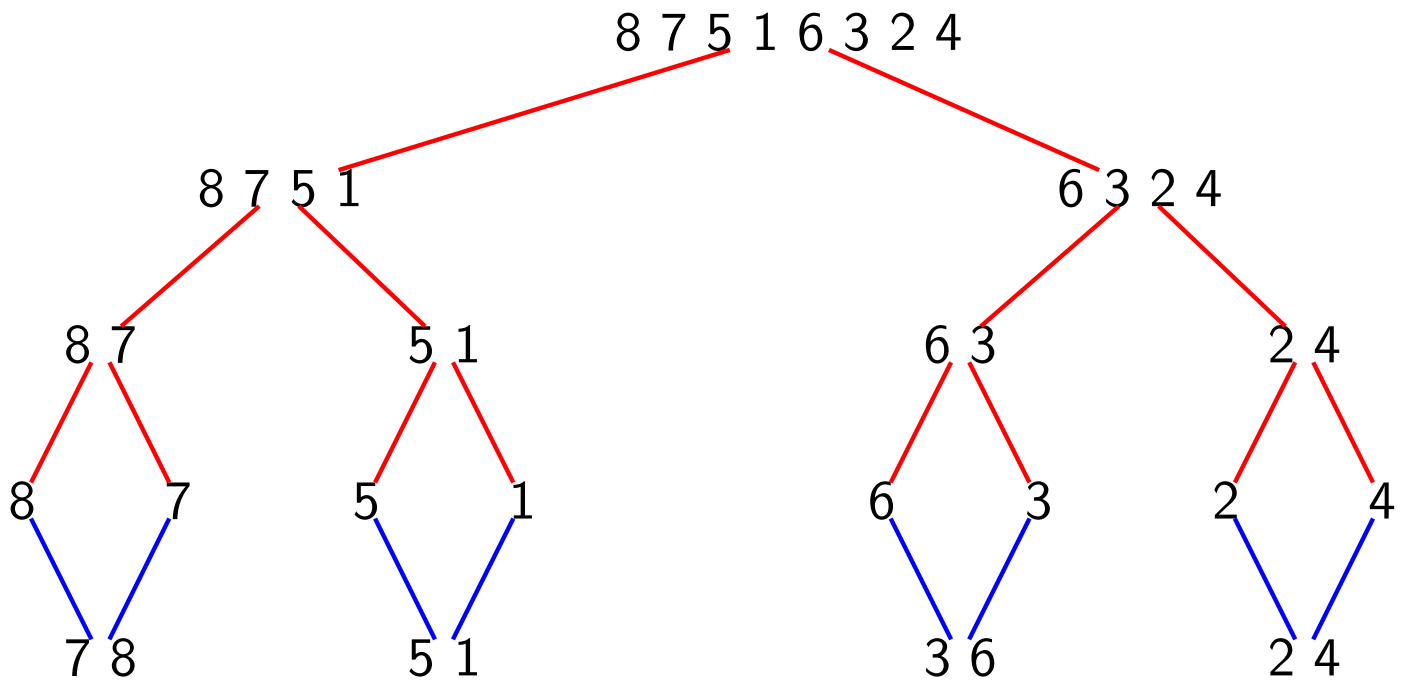

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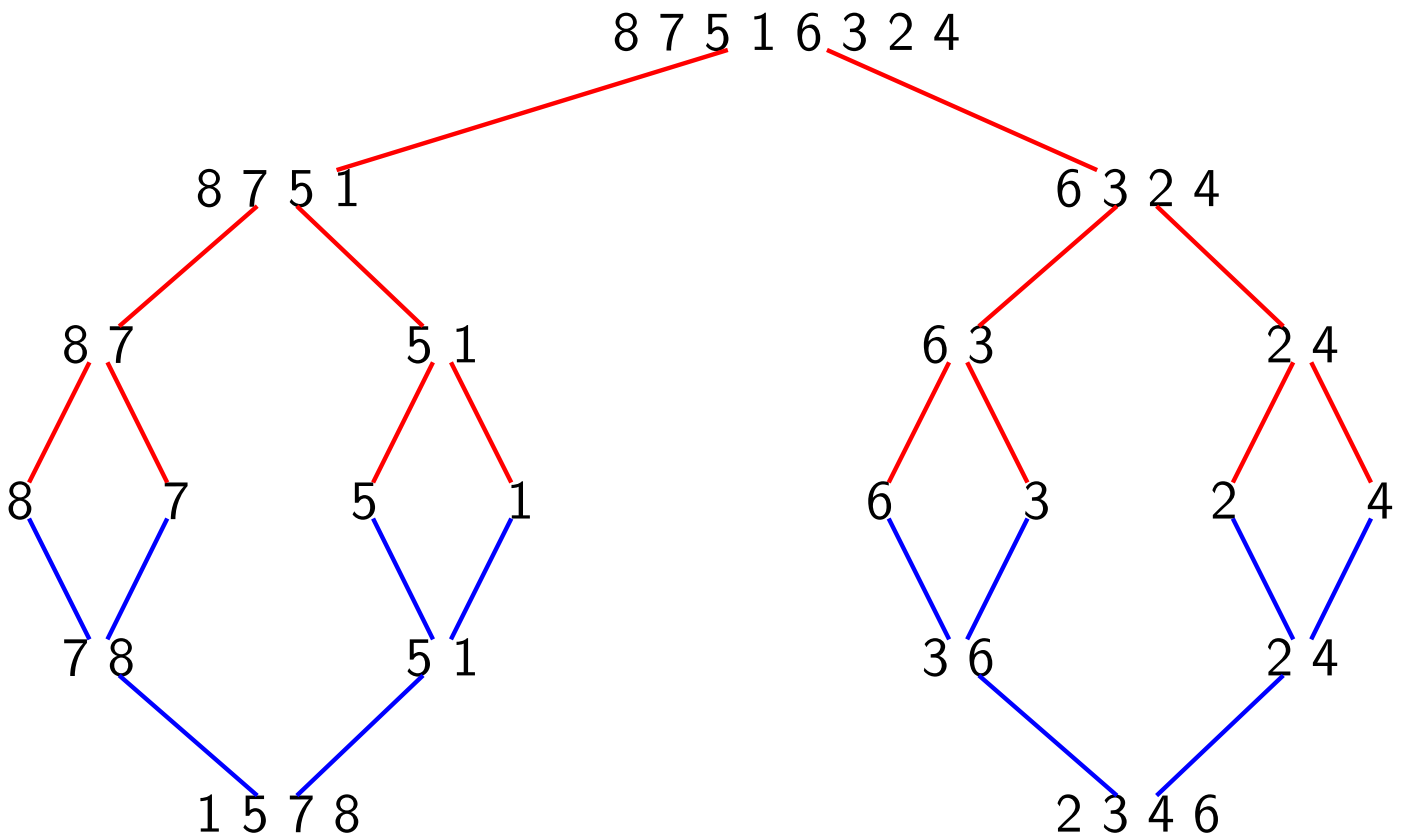

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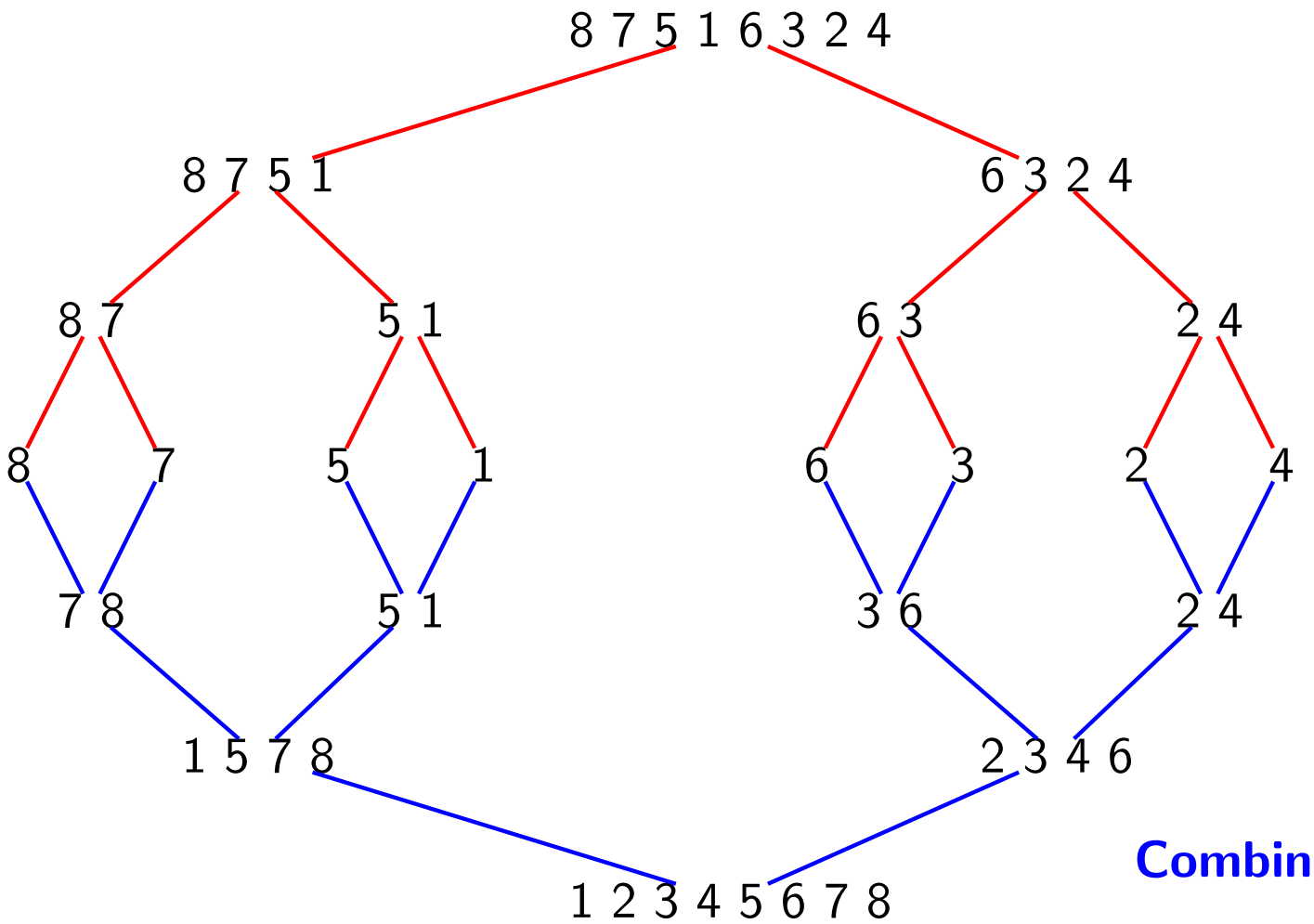

Combining...

Analyzing MergeSort

- We count key comparisons
- Assume `wlog` that n is a power of 2
- $C(n) = 2C(n/2) + M(n),\ n > 1, C(1) = 0$
- Applying the Mater Theorem:

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  - $C(n) \in \Theta(n \lg n)$

Specifying QuickSort

QuickSort($A[l \ldots r]$)

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if l > r
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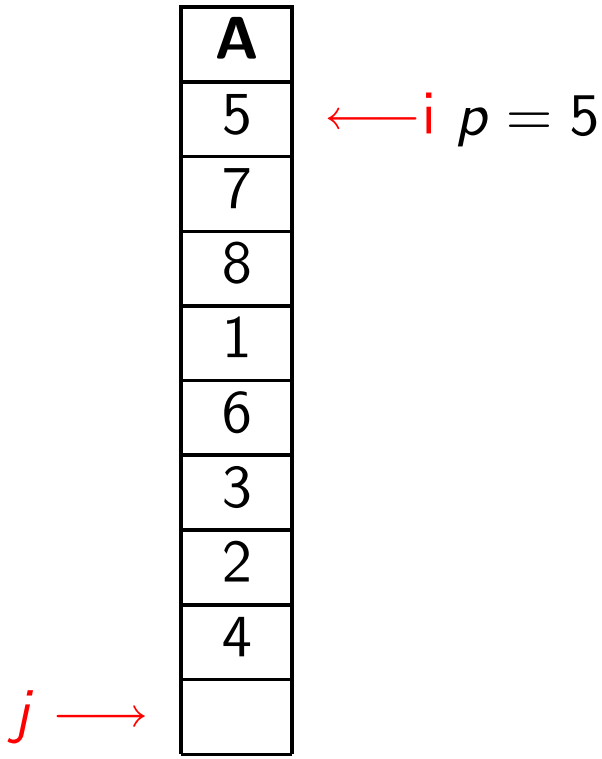
Partition($A[l \ldots r]$)

```
p ⟵ A[l]
i ⟵ l
j ⟵ r + 1
repeat
  repeat i++ until p ≥ A[i]
  repeat j−− until p ≤ A[j]
  Swap(A[i], A[j])
until i ≥ j
Swap(A[i], A[j])
Swap(A[l], A[j])
```

Partition



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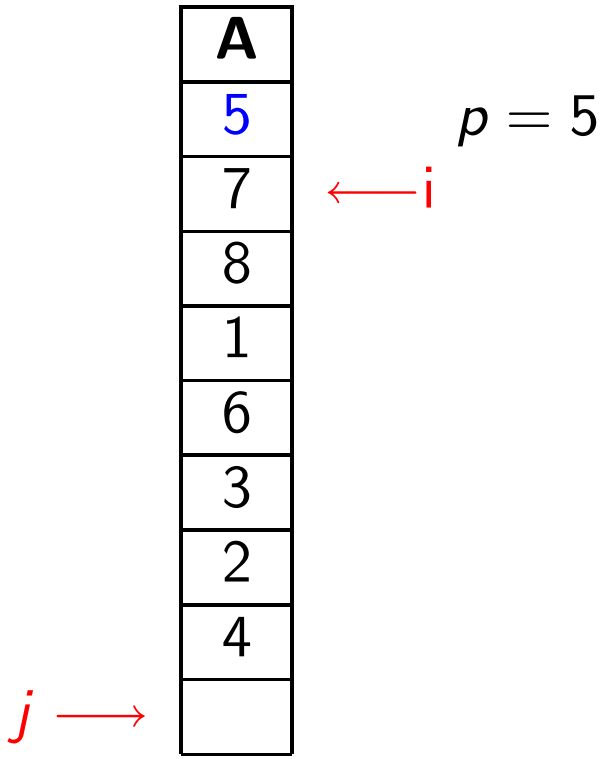

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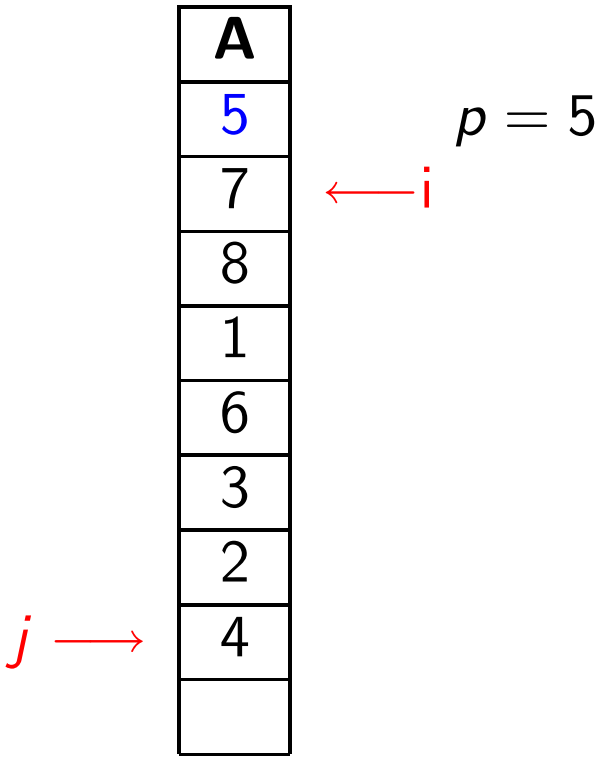

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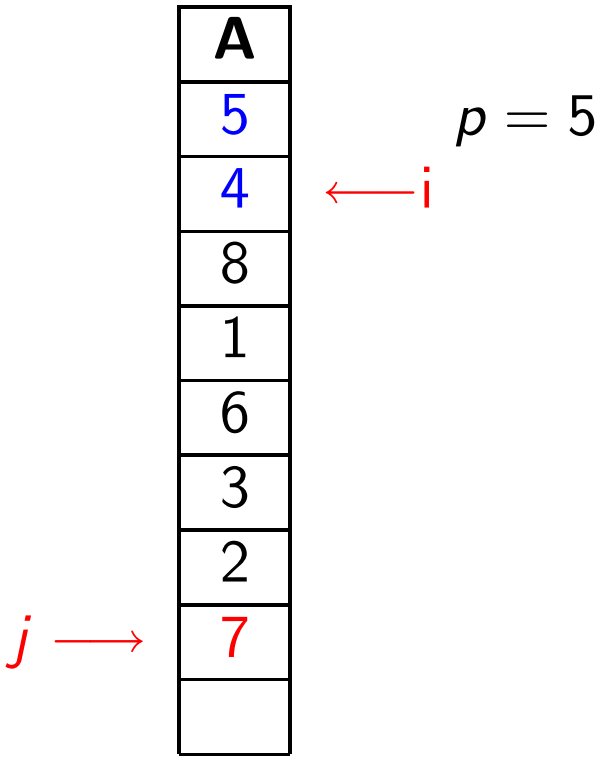

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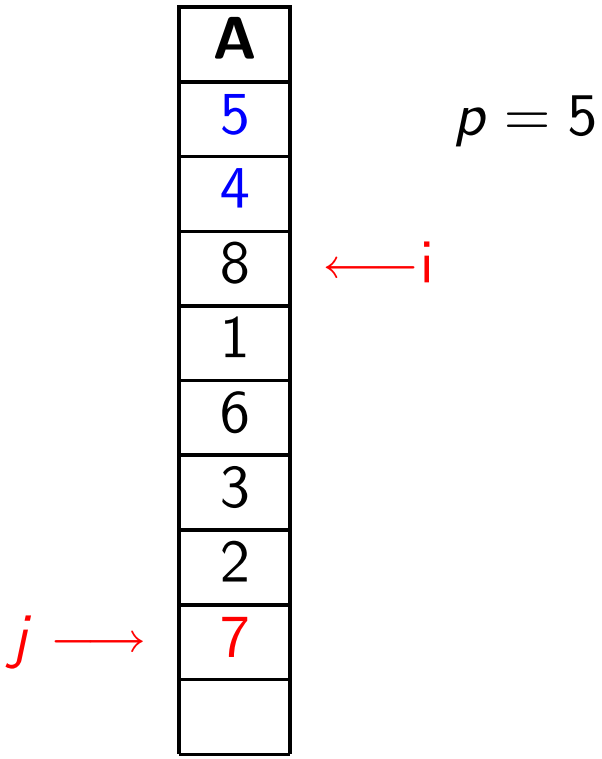

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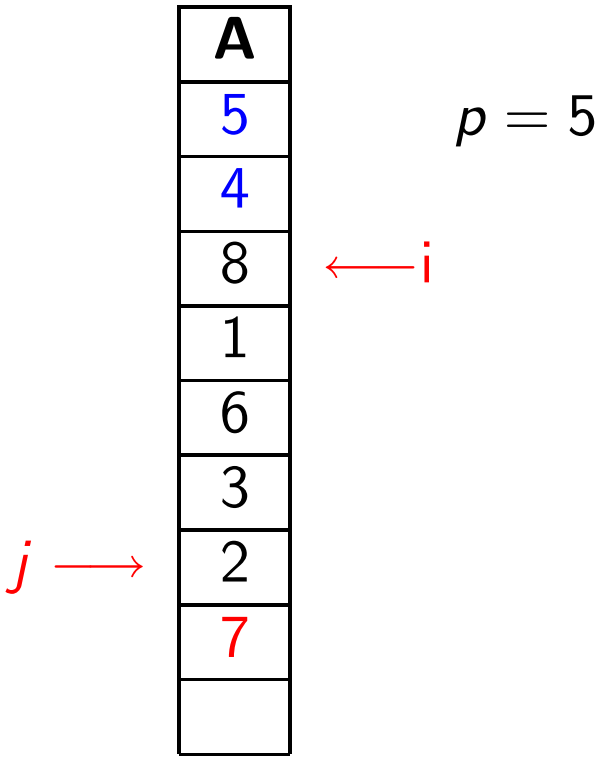

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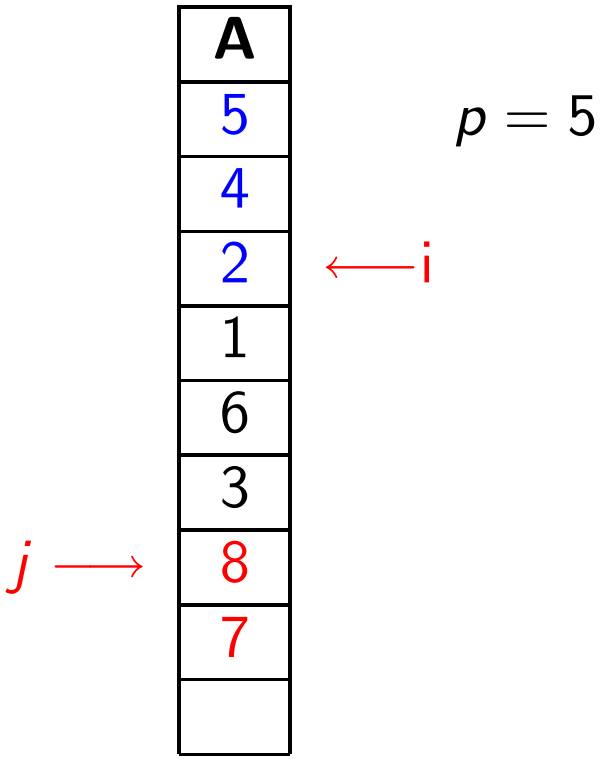

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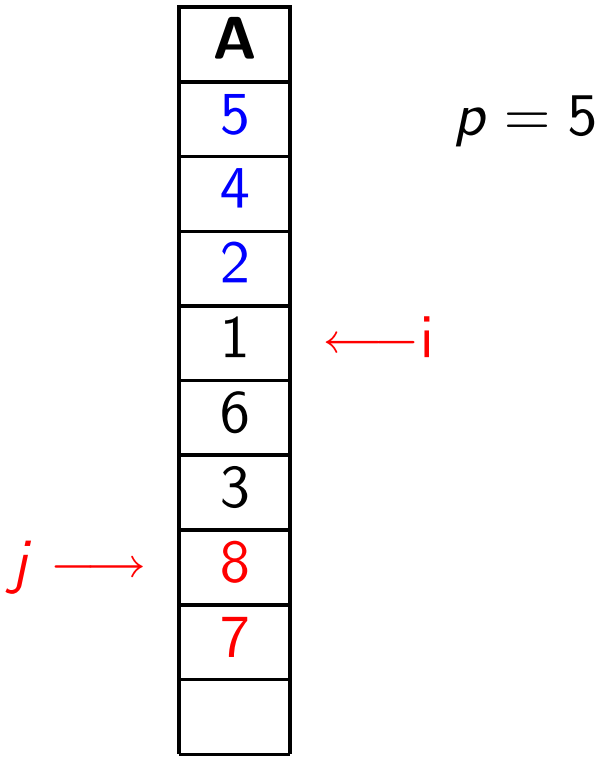

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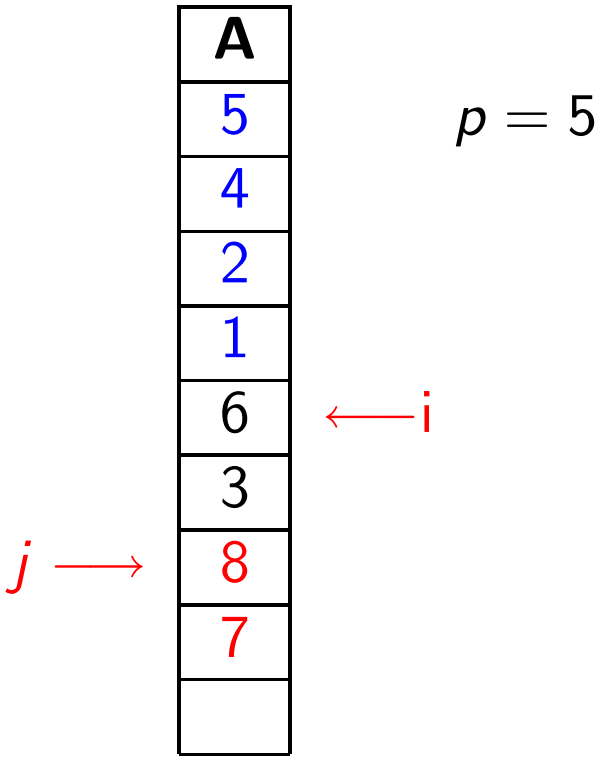

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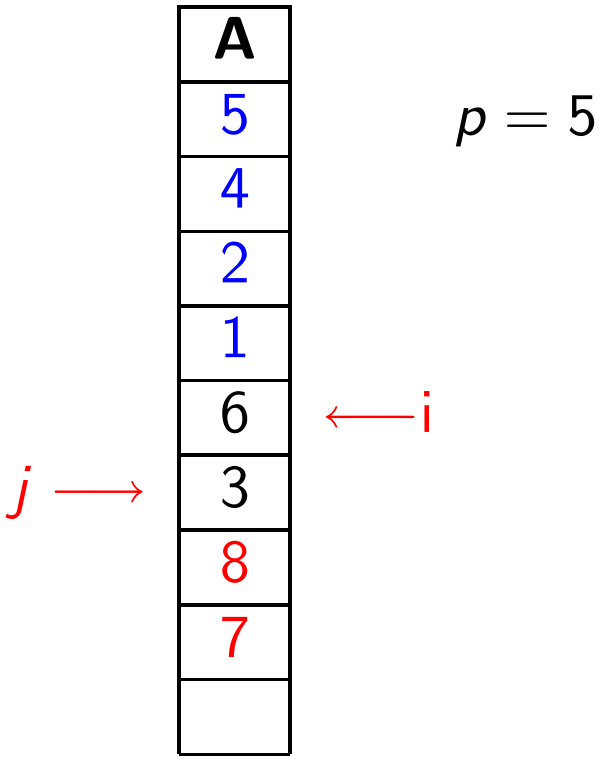
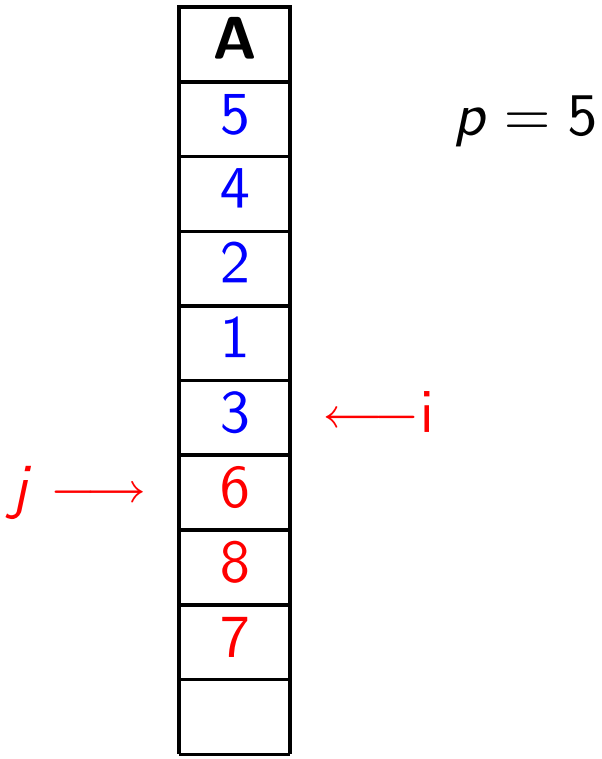


Partition



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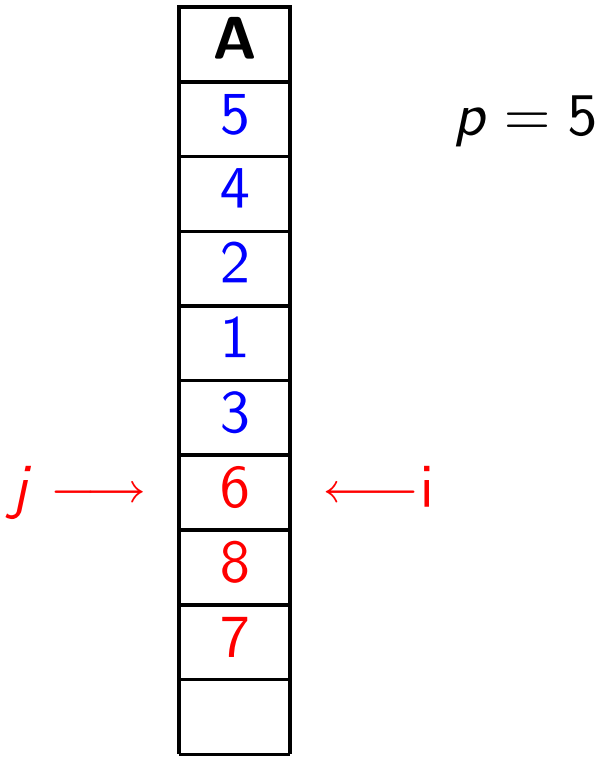

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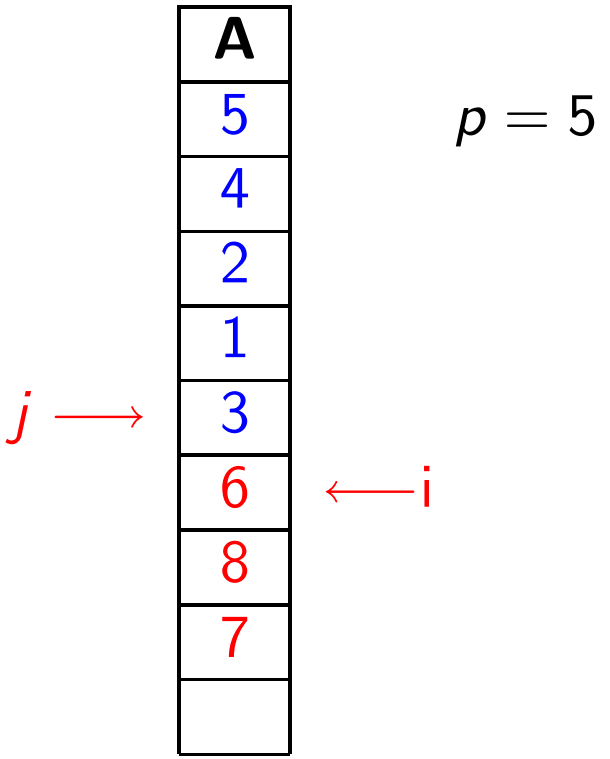

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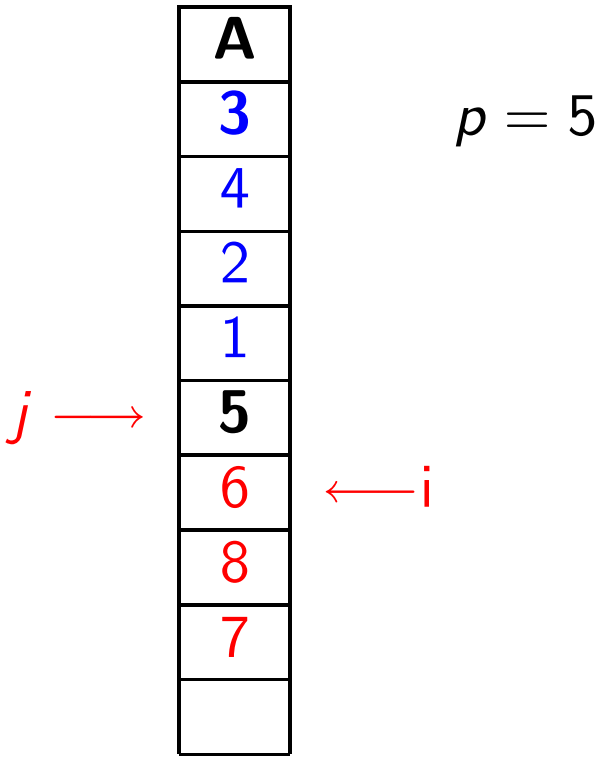

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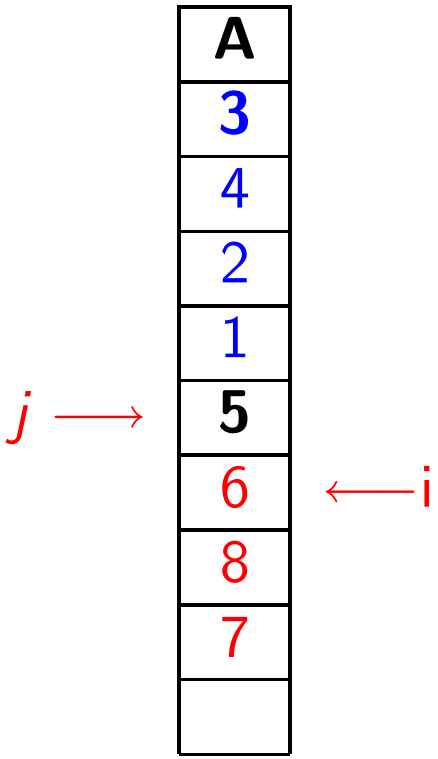
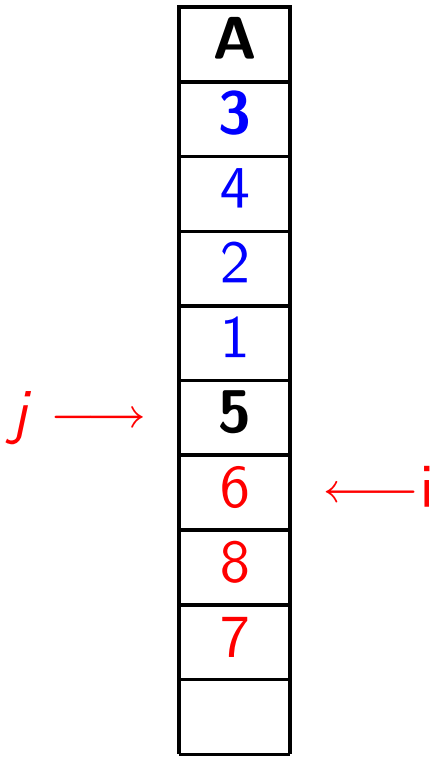


$p = 5$

How many comparisons?

Partition



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n

Decomposing & Combining

Decomposing...

5 7 8 1 6 3 2 4

Decomposing & Combining

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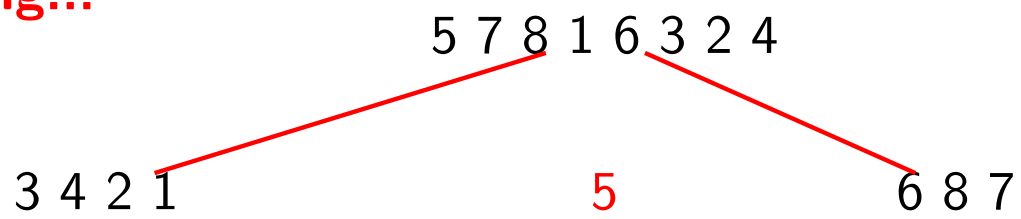

Decomposing & Combining

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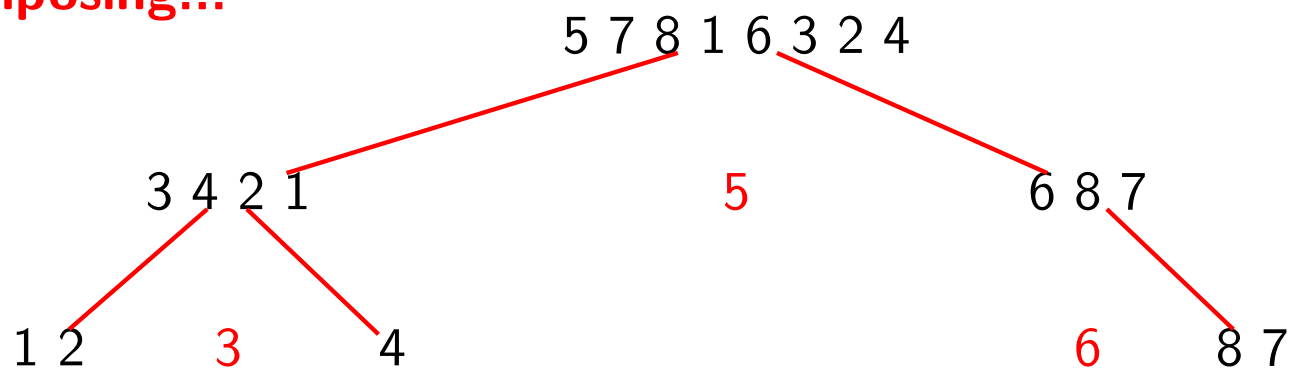

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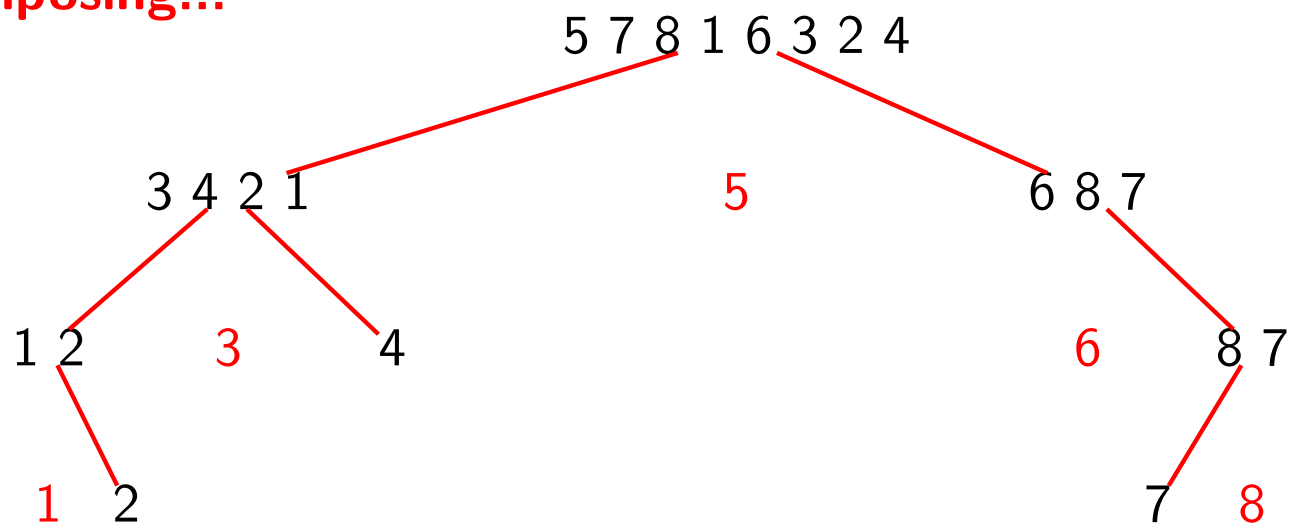

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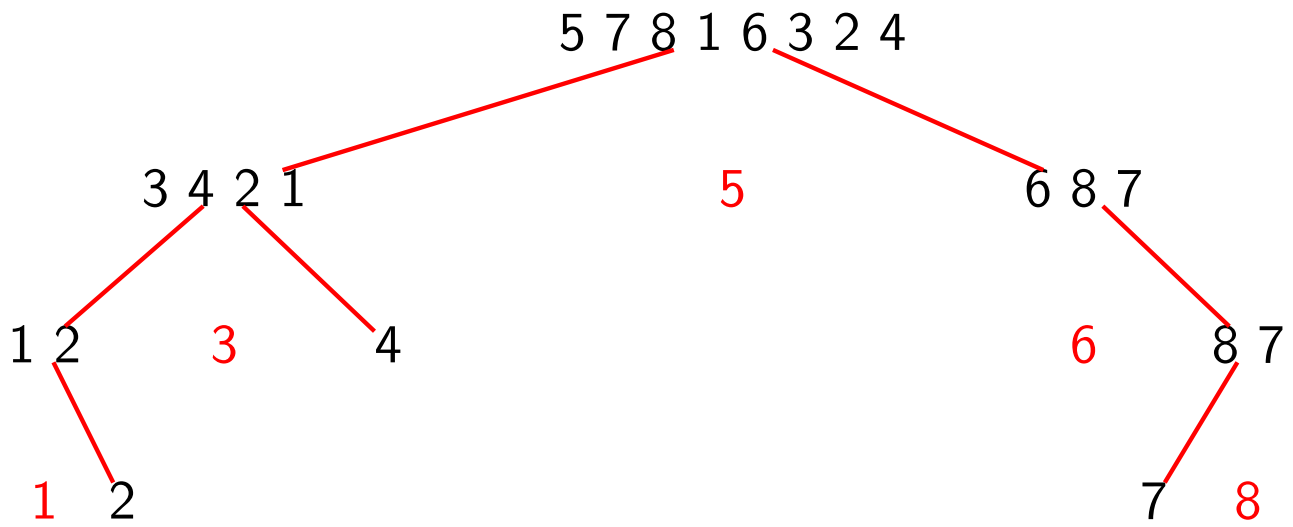
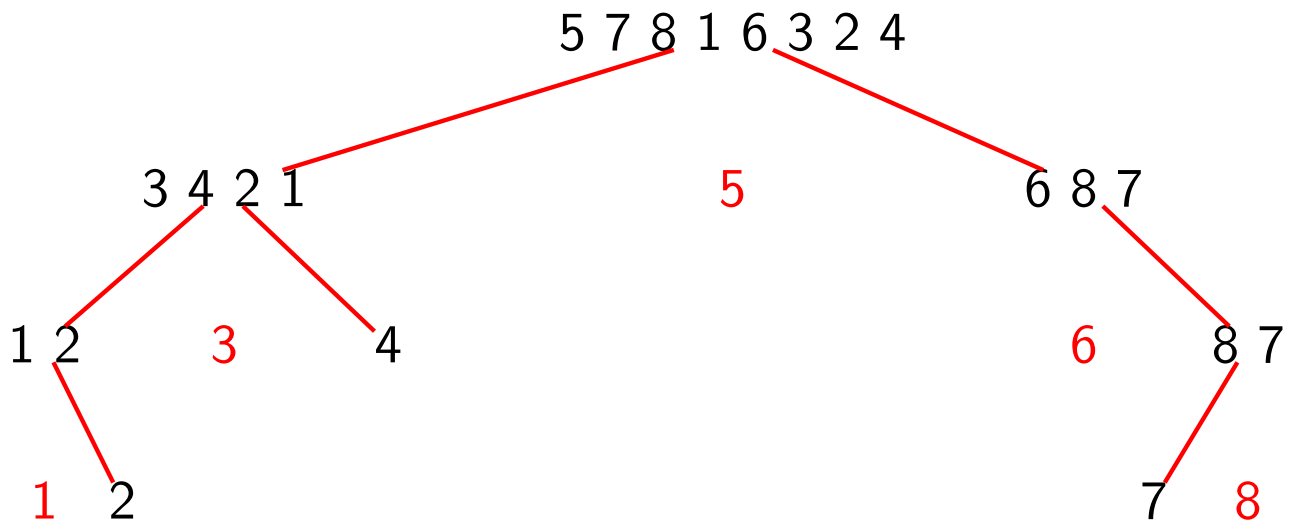


1 2 3 4 5 6 7 8

Combining...

Decomposing & Combining



1 2 3 4 5 6 7 8 **Combining...**

(It's already combined!)

Analyzing QuickSort

In QuickSort, the size of the split depends on the result of the Partition function...

- Best case:
- Worst case:
- Average case:

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  - $s \in [0, n-1], Pr\{s\} = \frac{1}{n} \forall s$
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Some fixes include:
Randomizing input order
median-of-three partitioning

Specifying BinarySearch

BinarySearch($A[0 \ldots n-1, K]$)

```
l ⟵ 0
r ⟵ n − 1
while l ≤ r do
    m ⟵ ⌊(l+r)/2⌋
    if K = A[m] return m
    else if K < A[m] r ⟵ m − 1
    else l ⟵ m + 1
return −1
```

$K = 51$

	A	
l ⟶	0	17
	1	22
	2	30
	3	35
	4	37
⊢m	5	43
	6	51
	7	64
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- Best case:
  - Key is at the midpoint in the list
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  - *Very* unlikely...
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- Average case:

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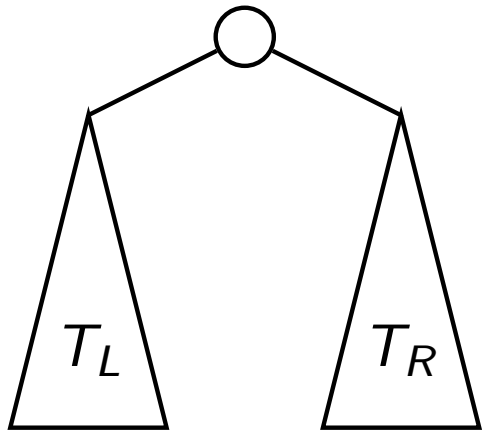
- Average case:
  - Not substantially worse than the worst case, actually
  - $C_{avg} \in \Theta(\lg n)$

Binary Trees

- *Binary tree*, T — a finite set of nodes that is either empty or consists of a root and two disjoint binary trees T_L and T_R, called the *left* and *right* subtree, respectively
- Note that the very definition recursively divides the tree into smaller, similar structures
- Many tree-related problems are solved by applying D&C methods
- In particular, many tree-related problems require an algorithm to *traverse* a tree



The Height Algorithm

Height(T)

```
if T = ∅ return -1
else return max{Height(T_L), Height(T_R)} + 1
```

- Measure problem size by the number of nodes in a given tree, $n(T)$
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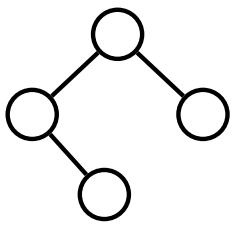

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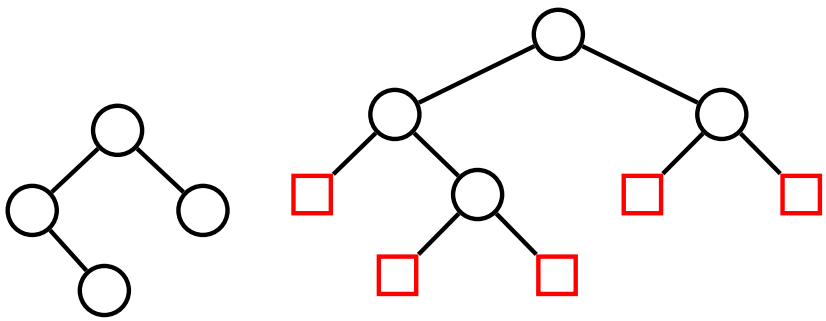

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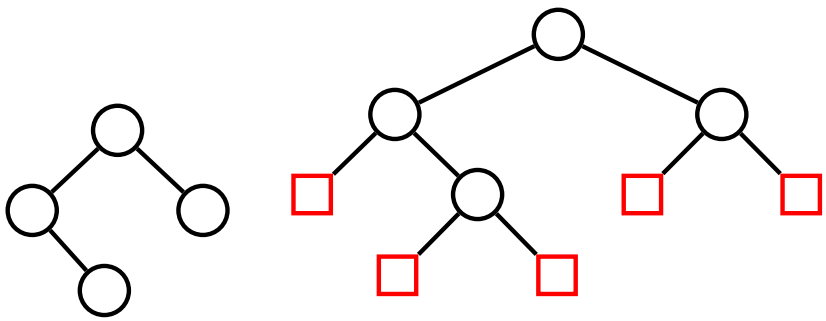

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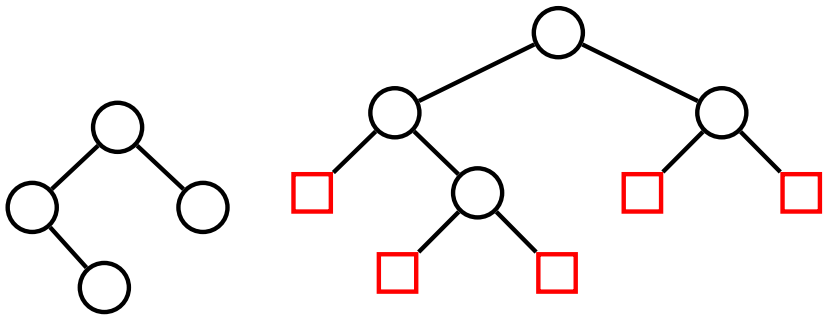

How many external nodes does a tree with n internal nodes have?

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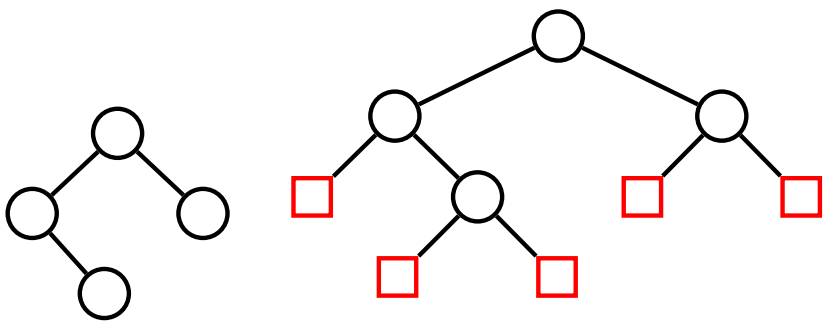

How many external nodes does a tree with n internal nodes have? $x = n + 1$

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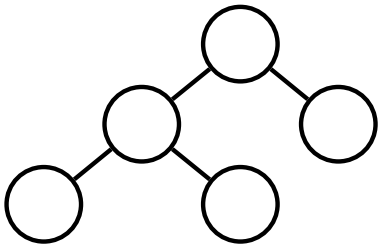
Different Kinds of Tree Traversal

Preorder traversal:
Visit root, then left subtree, then right subtree



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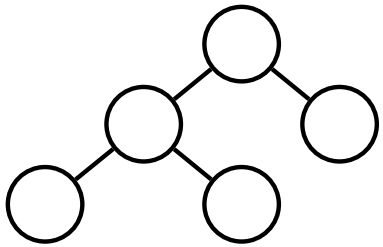

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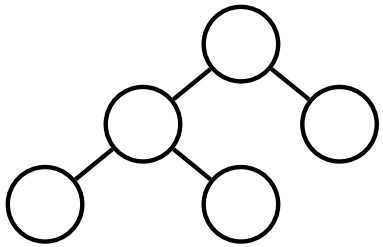

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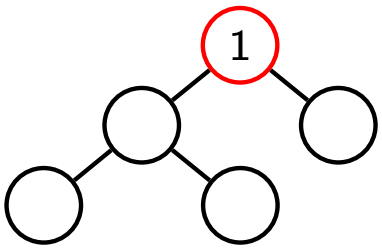

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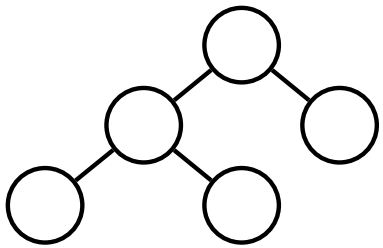

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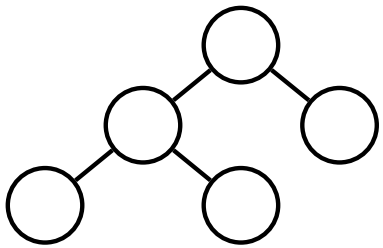

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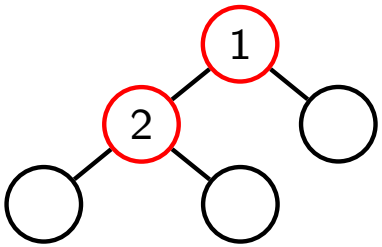

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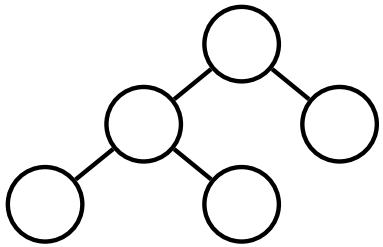

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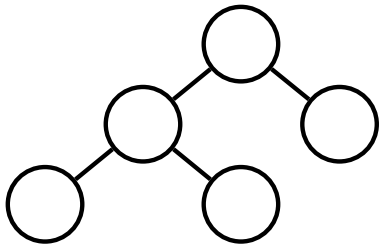

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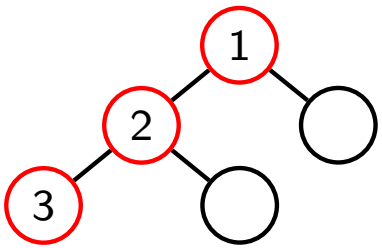

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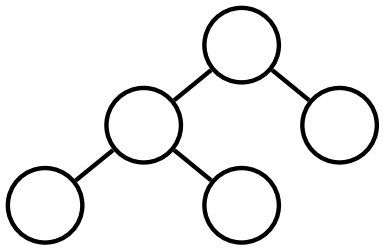

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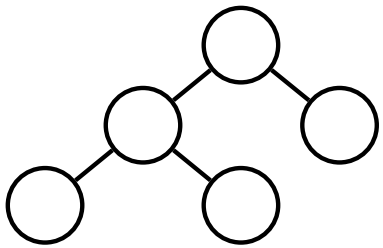

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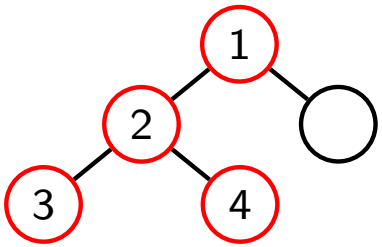

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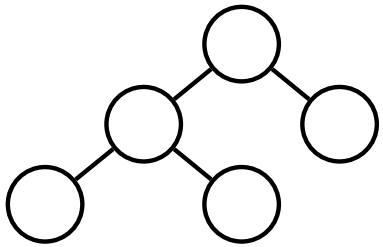

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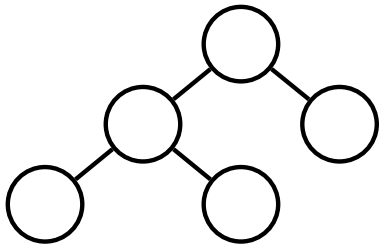

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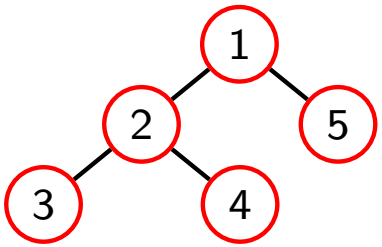

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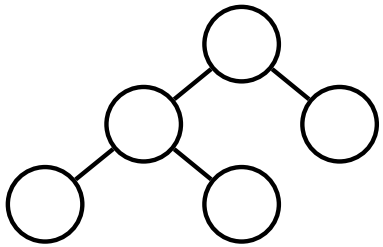

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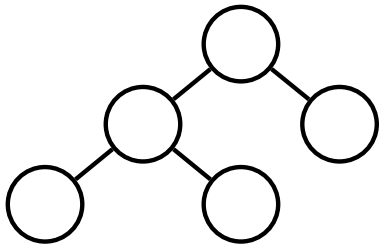

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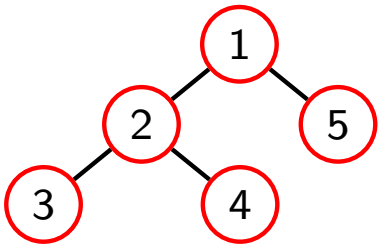

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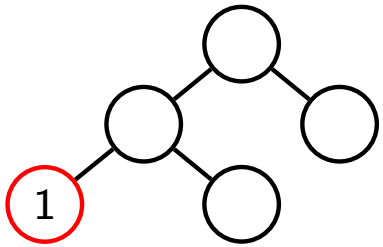

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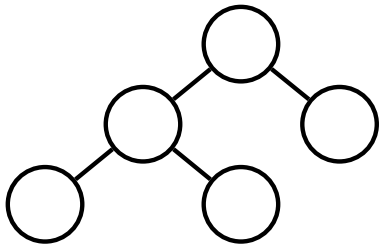

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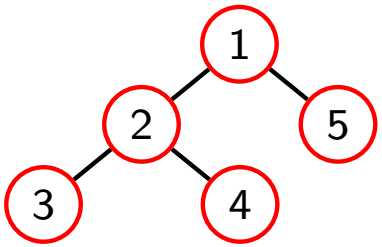

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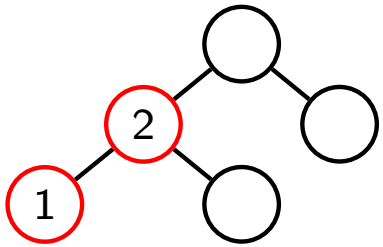

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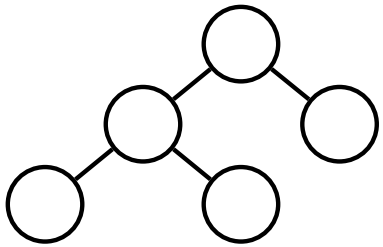

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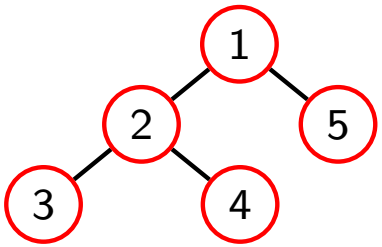

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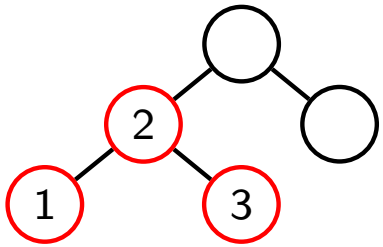

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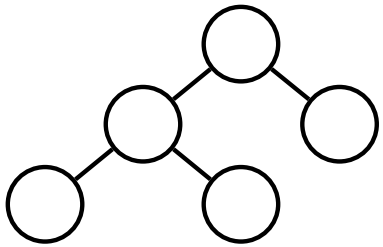

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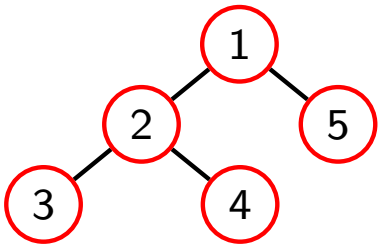

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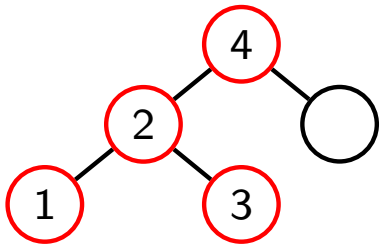

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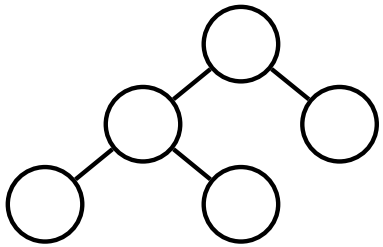

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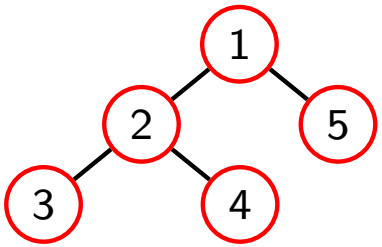

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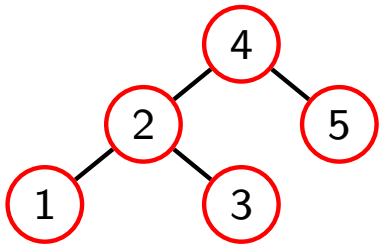

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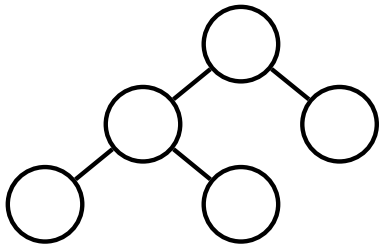

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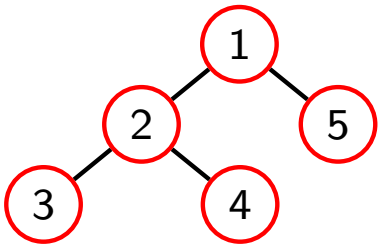

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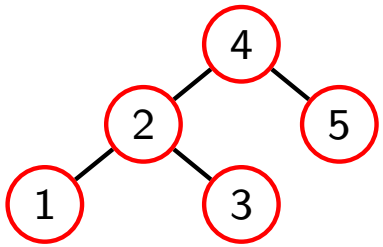

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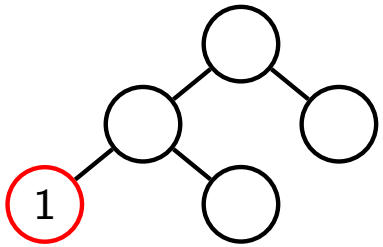

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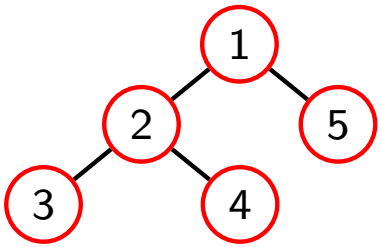

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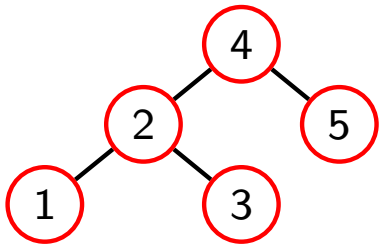

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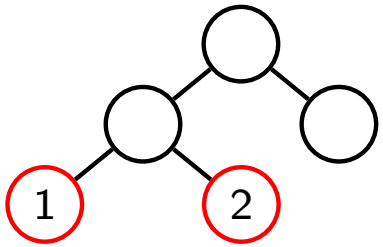

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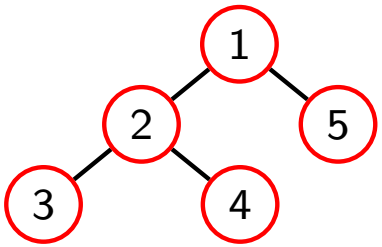

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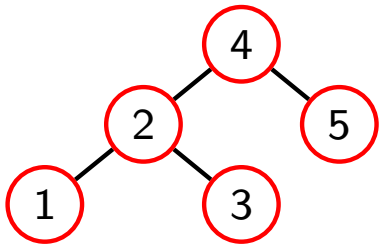

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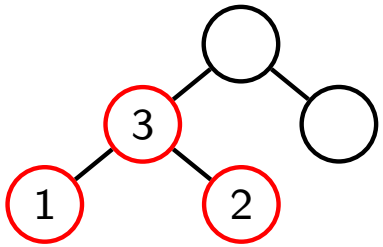

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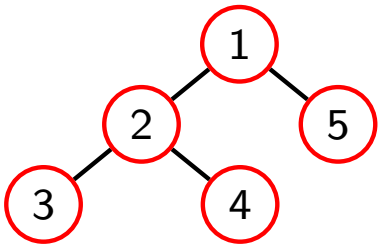

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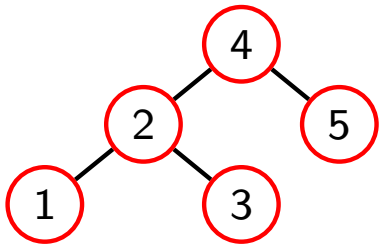

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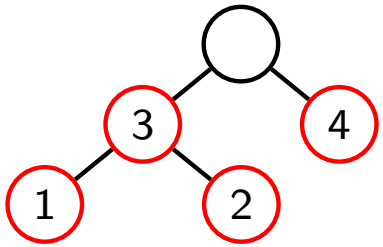

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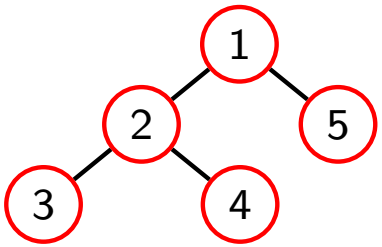

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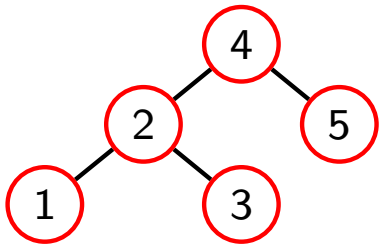

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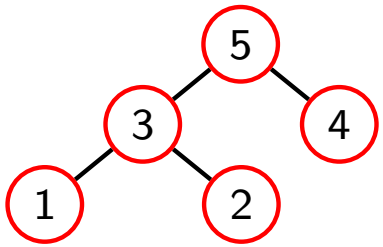

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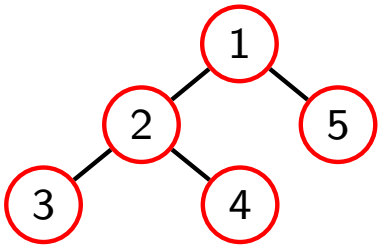

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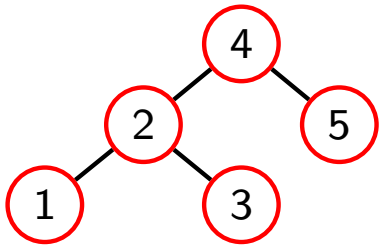

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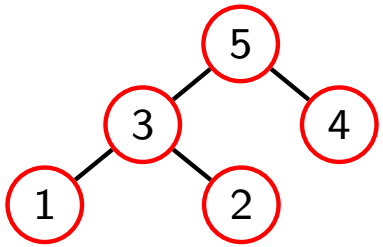

In general, traversals are $\Theta(n)$, but not all binary tree operations require full traversal of the tree (e.g., Find, Insert, etc.)

Multiplication of Two-Digit Integers

- Pen-and-paper multiplication:
  - If there are n digits in first integer, m in the second, this requires nm digit multiplications
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- For any pair of two-digit numbers ($a = a_1 a_0$, $b = b_1 b_0$):

$$\begin{aligned} c &= a \cdot b = c_2 \cdot 10^2 + c_1 \cdot 10^1 + c_0, \text{ where} \\ c_2 &= a_1 \cdot b_1 \\ c_0 &= a_0 \cdot b_0 \\ c_1 &= (a_1 + a_0) \cdot (b_1 + b_0) - (c_2 + c_0) \end{aligned}$$

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\end{aligned}
$$

We saved one multiplication!

Multiplication of Large Integers

- We can generalize this method using D&C
- Let the digits of a and b be partitioned `s.t.`:
  - $a = a_1 a_0 \Longrightarrow a = a_1 \cdot 10^{n/2} + a_0$
  - $b = b_1 b_0 \Longrightarrow b = b_1 \cdot 10^{n/2} + b_0$
- Using the same trick:

$$\begin{aligned} c &= a \cdot b = \left(a_1 \cdot 10^{n/2} + a_0\right) \cdot \left(b_1 \cdot 10^{n/2} + b_0\right) \\ &= (a_1 \cdot b_1) \cdot 10^n + (a_1 \cdot b_0 + a_0 \cdot b_1) \cdot 10^{n/2} + (a_0 \cdot b_0) \\ &= c_2 \cdot 10^n + c_1 \cdot 10^{n/2} + c_0, \text{ where} \\ c_2 &= a_1 \cdot b_1 \\ c_0 &= a_0 \cdot b_0 \\ c_1 &= (a_1 + a_0) \cdot (b_1 + b_0) - (c_2 + c_0) \end{aligned}$$

Divide-and-Conquer Multiplication

- Analysis of this is straightforward:
  - We count multiplications
  - We divide digits in half each time, apply multiplication three times when combining
  - Recurrence: $M(n) = 3M\left(\frac{n}{2}\right)$ for $n > 1,\ M(1) = 1$
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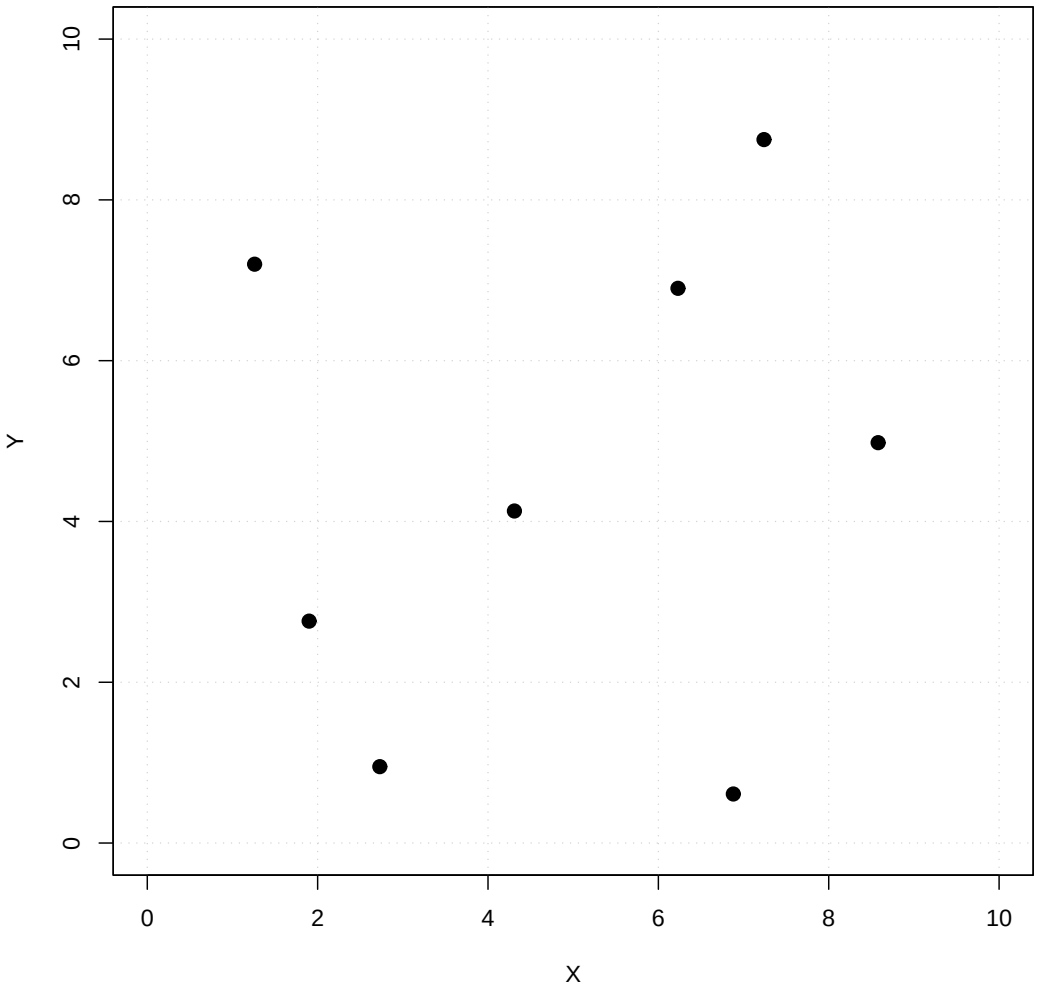
- For small to moderate sized input, the standard multiplication method is faster
- Similar methods can be applied for matrix multiplication (Strassen's Method, in book)
  - Use algebra to reduce (by one) the eight multiplications performed when working with two 2×2 matrices
  - Accrue 18 addition operations, whereas traditional matrix multiplication requires only 4
  - Partition complete matrix into four submatrices, recursively apply method
  - Count additions
  - Recurrence: $A(n) = 7A\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2$ for $n > 1,\ A(1) = 0$
  - $A(n) \in \Theta\left(n^{\lg 7} \approx n^{2.8}\right)$

D&C and the Closest Pair Problem



- Order the points by x-axis
- Recursively partition points
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- Pass the winning pair up

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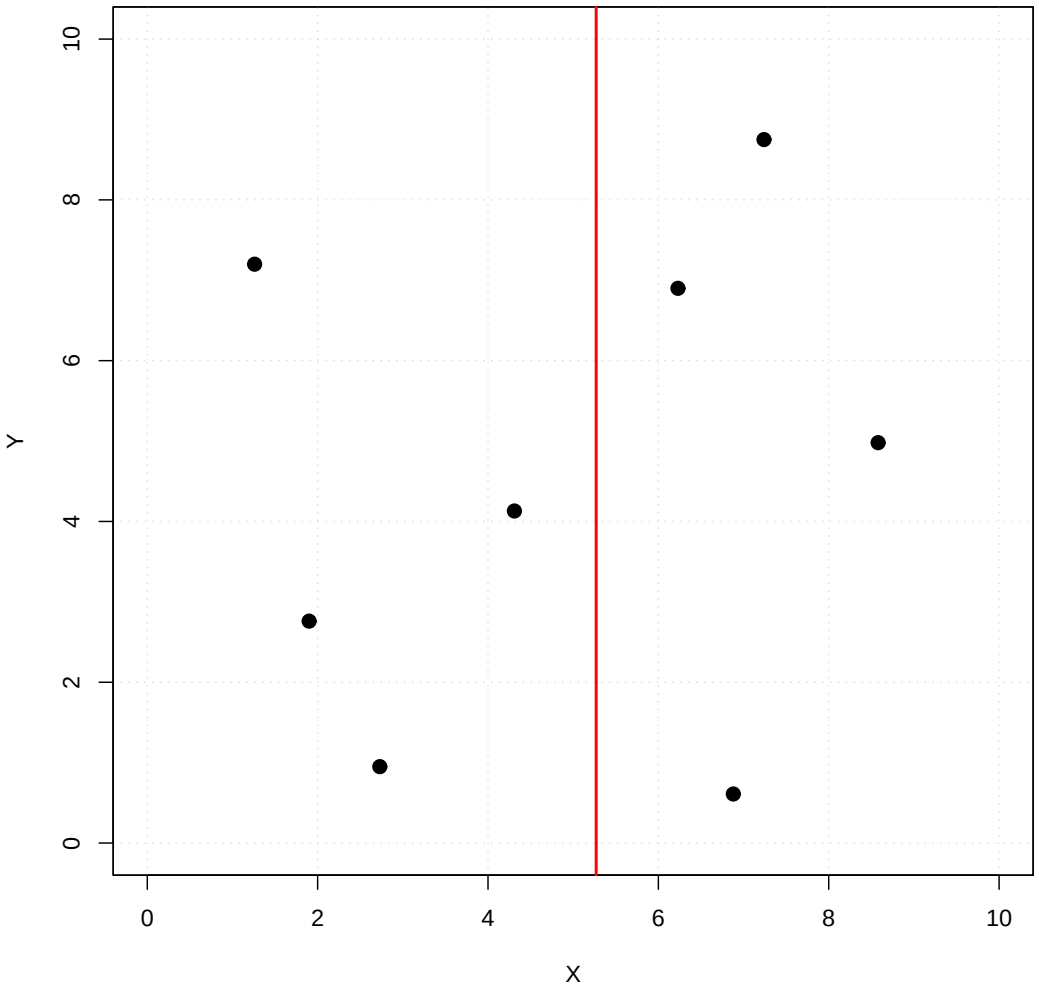

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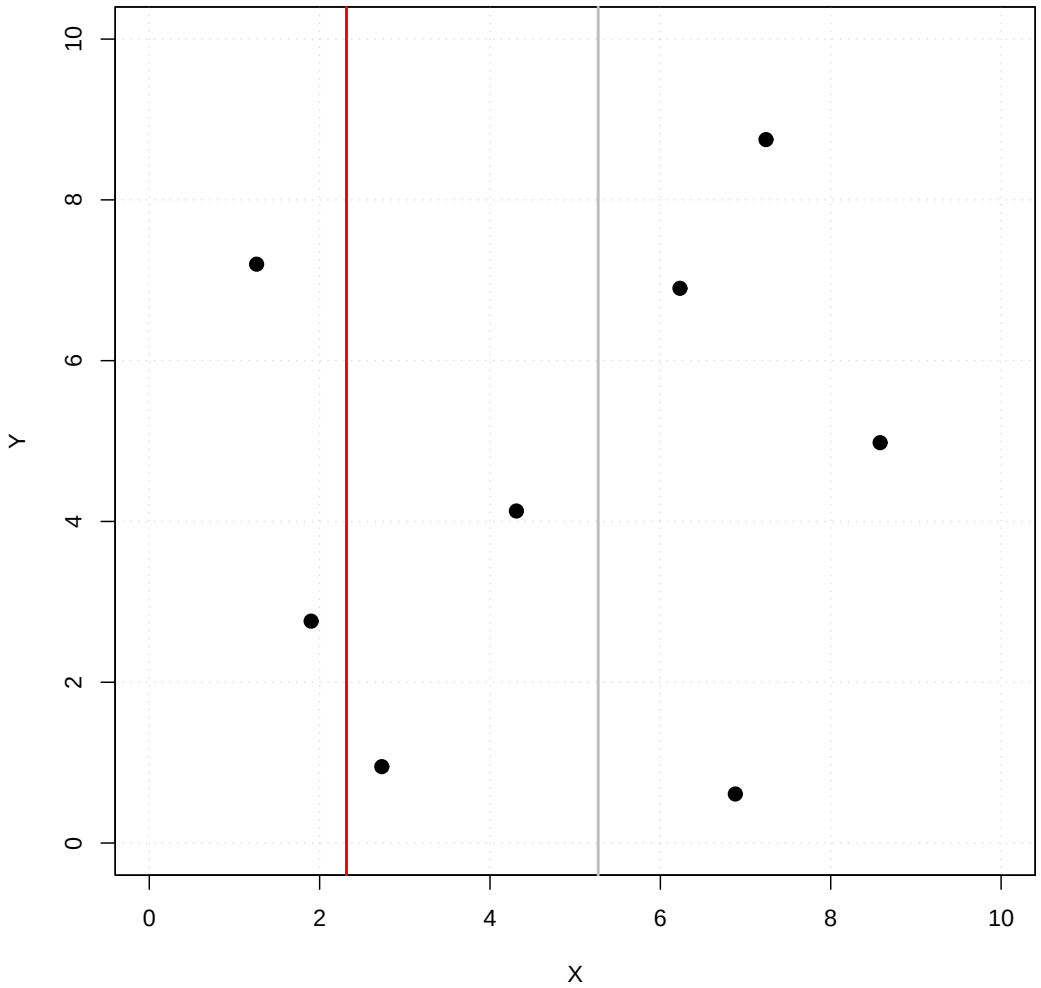

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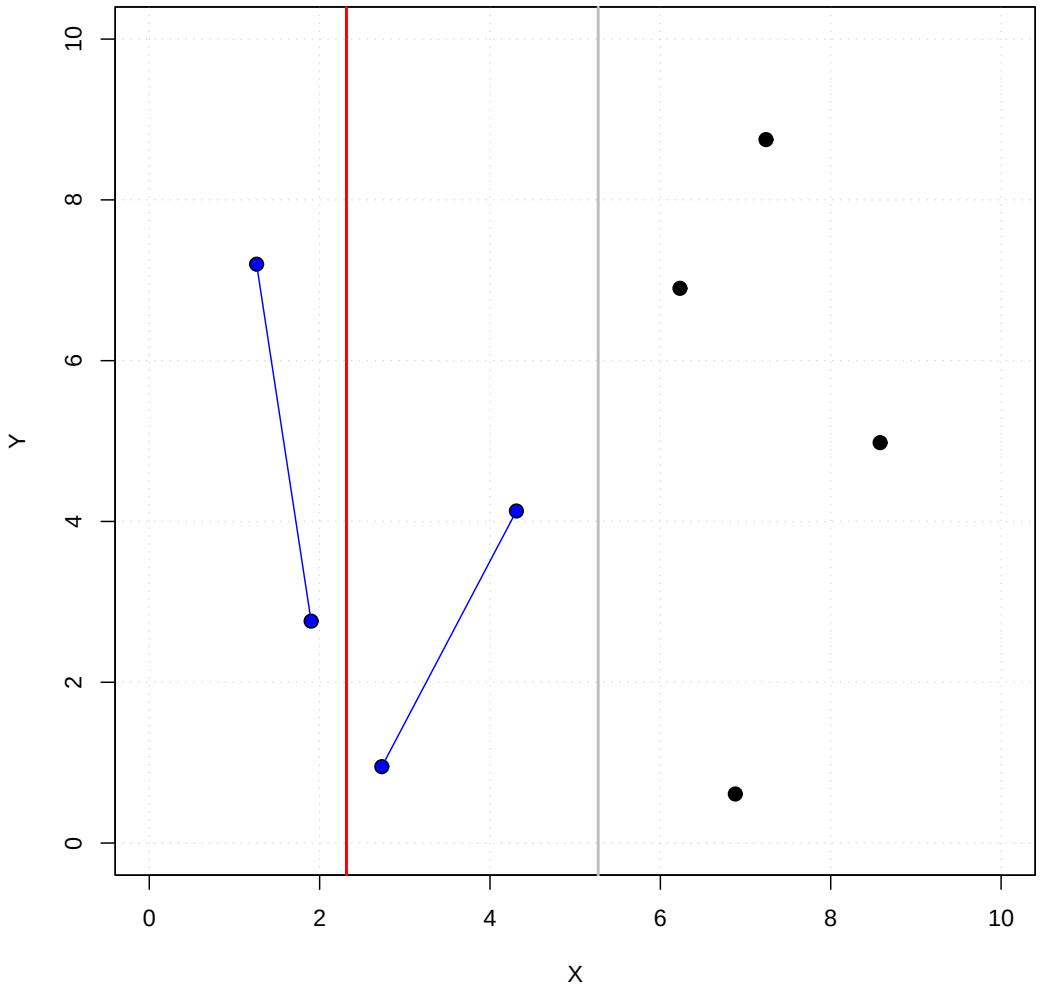

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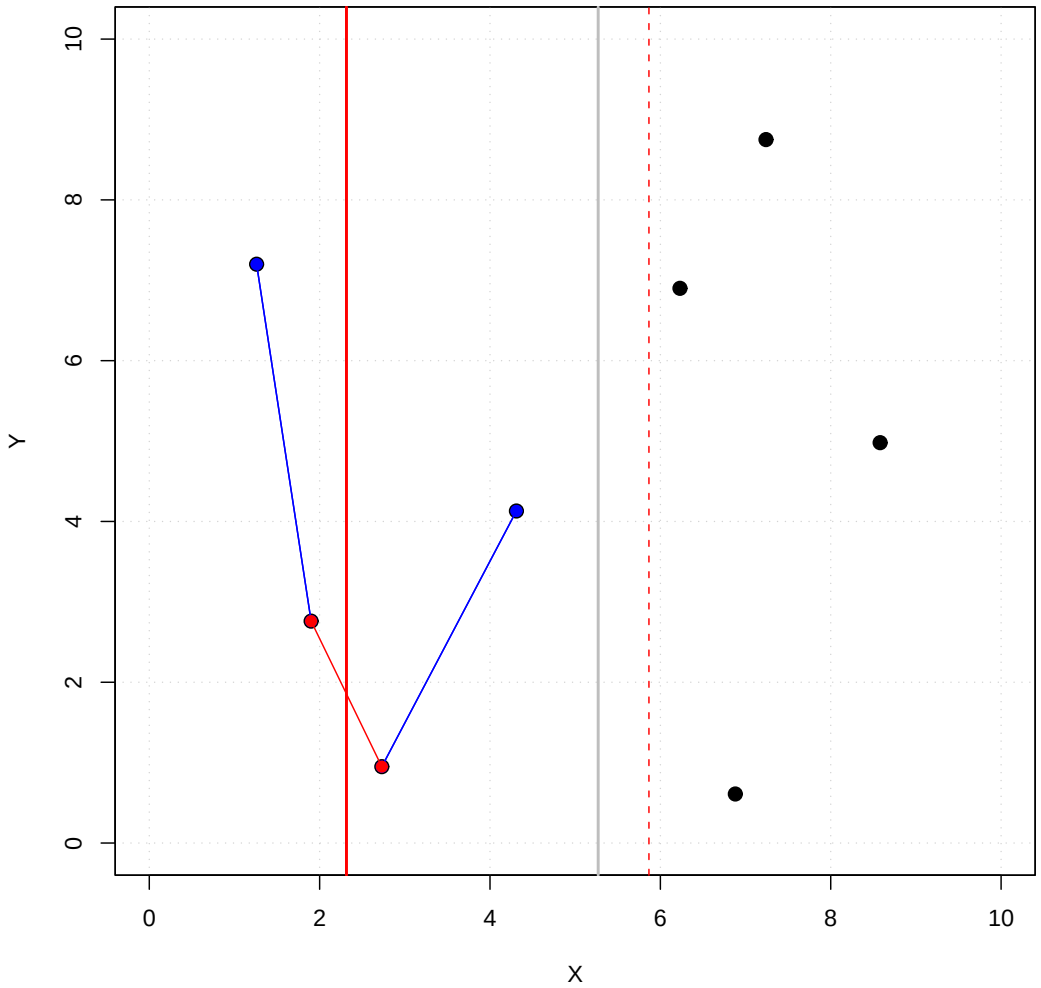

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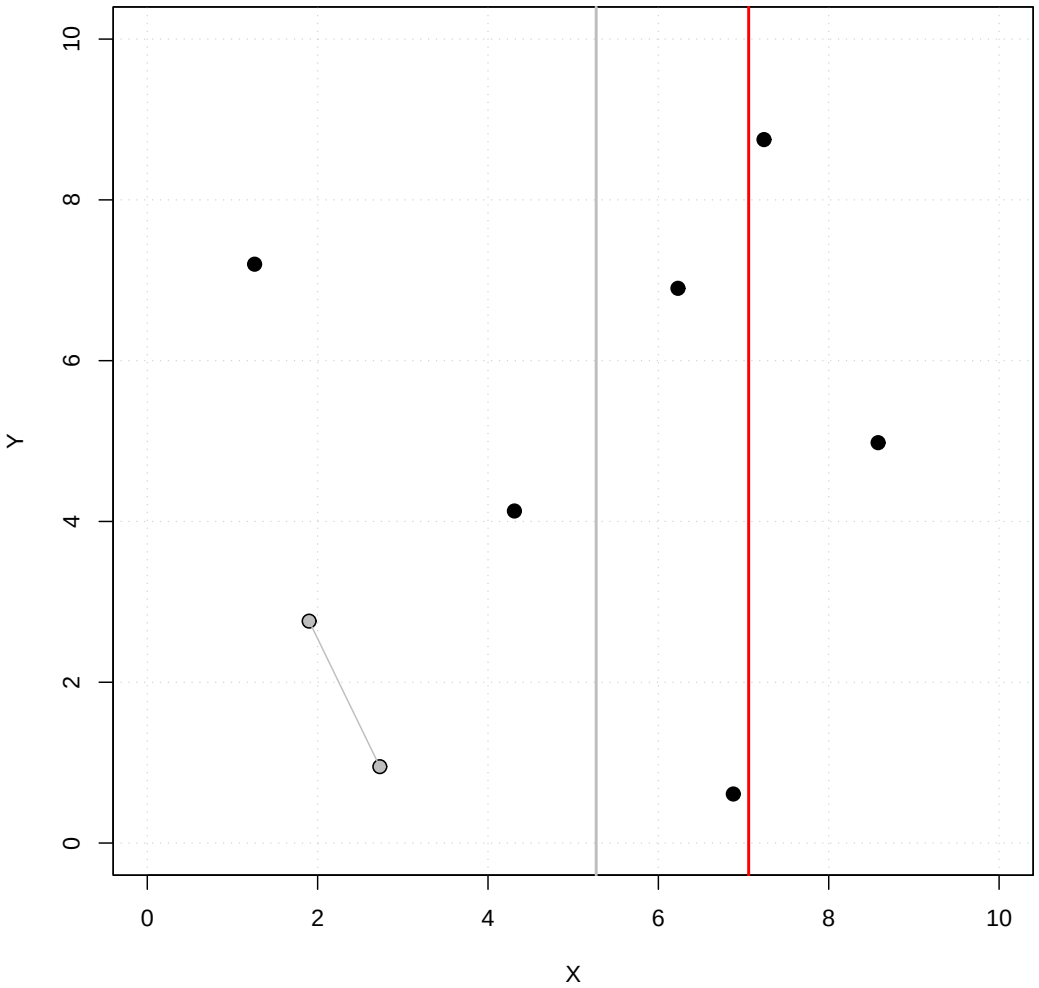

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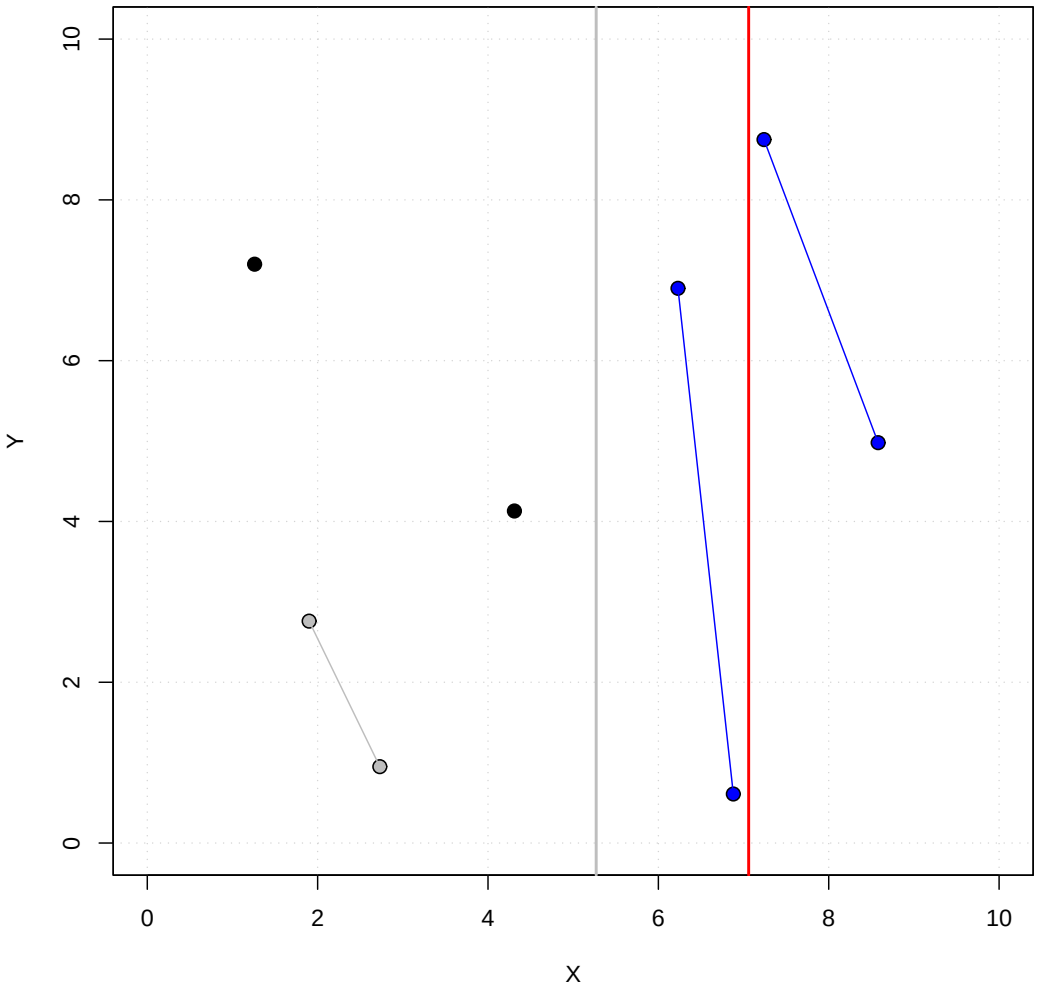

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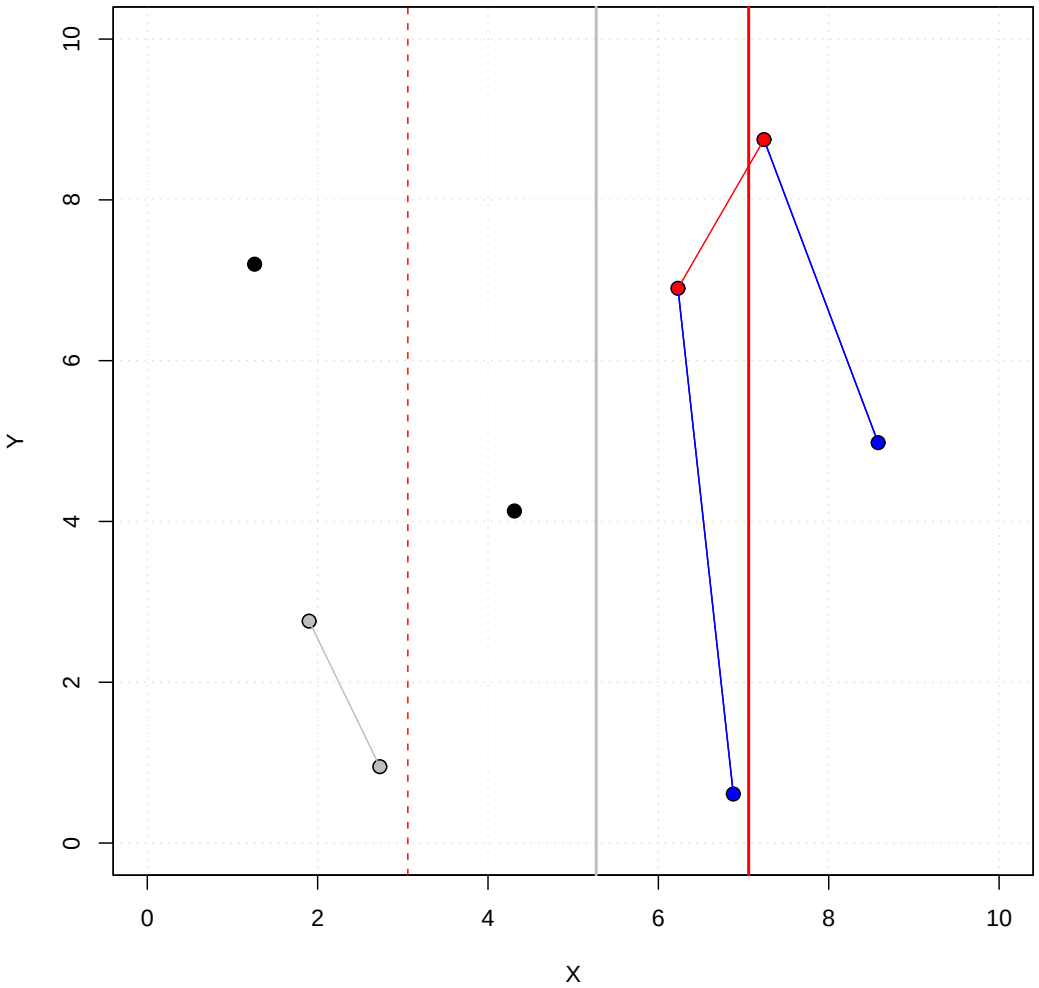

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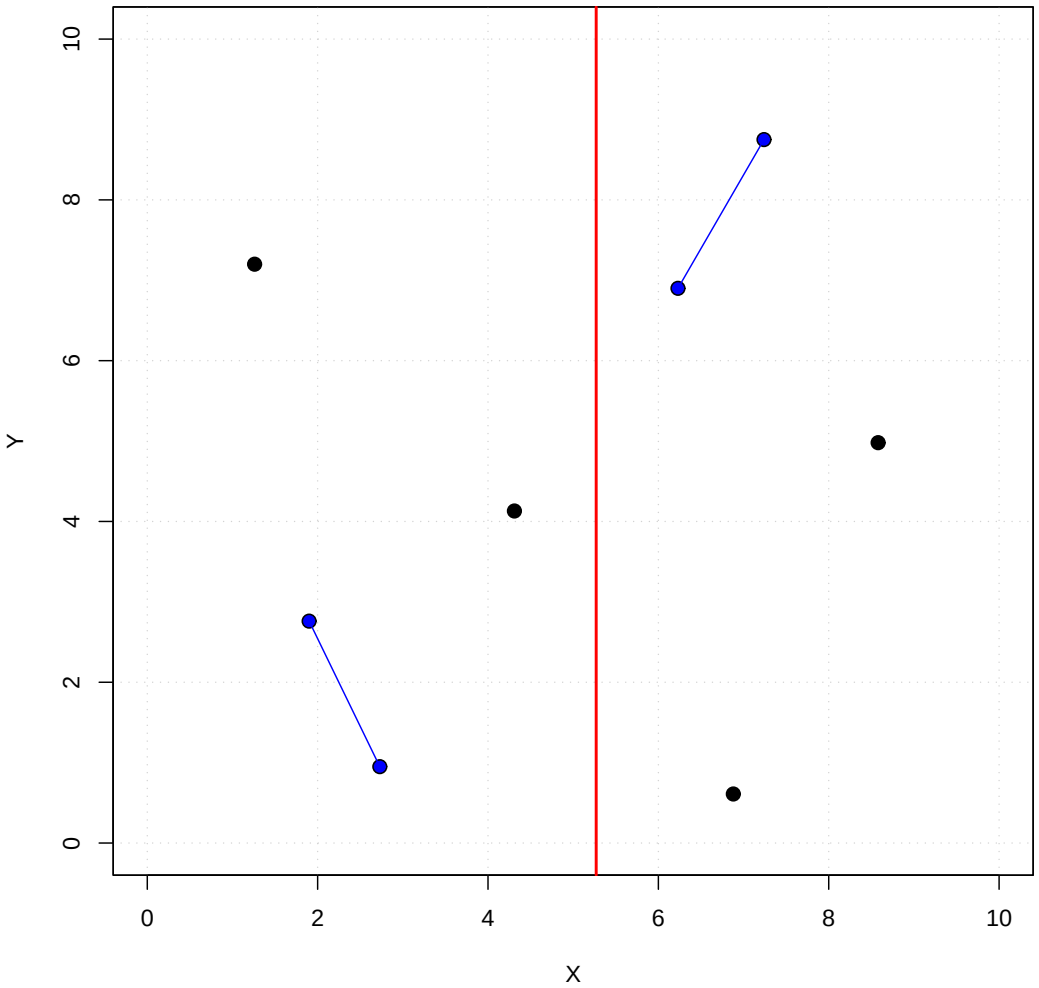

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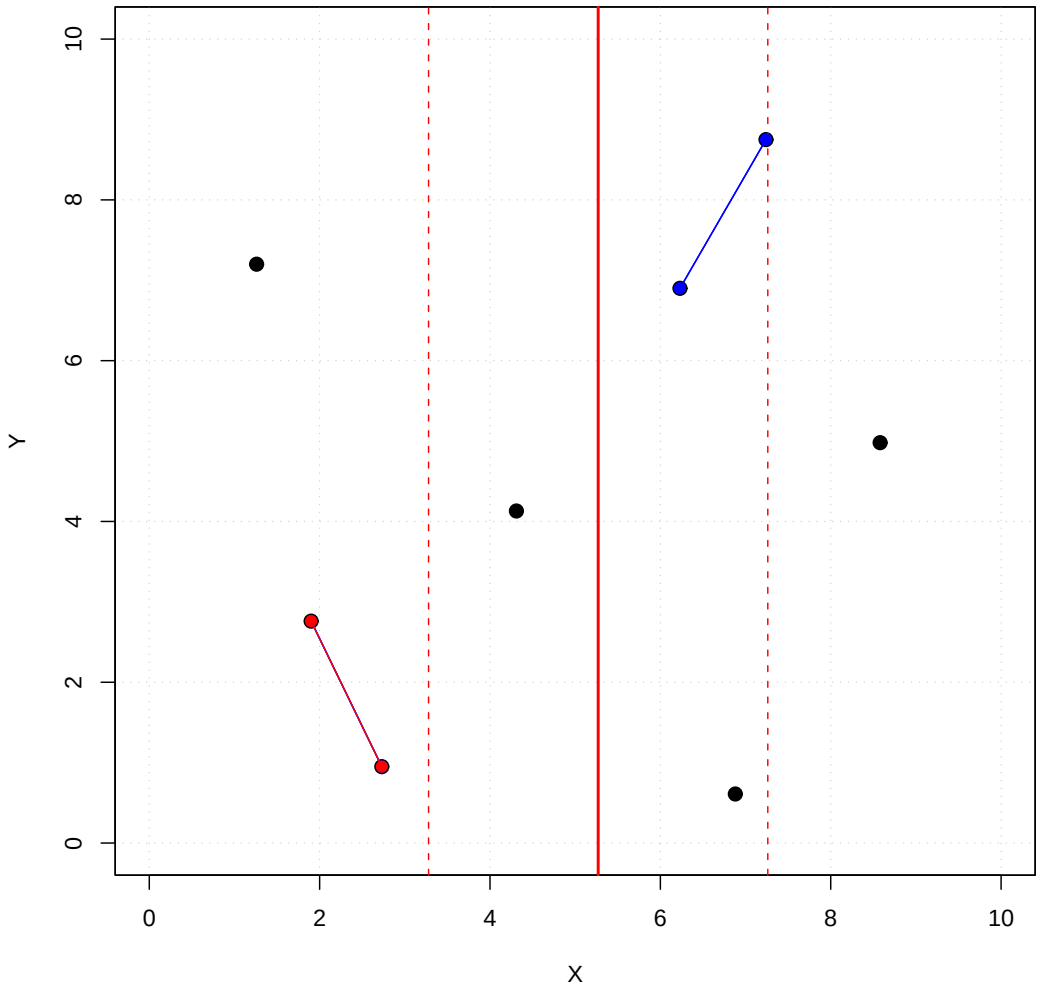

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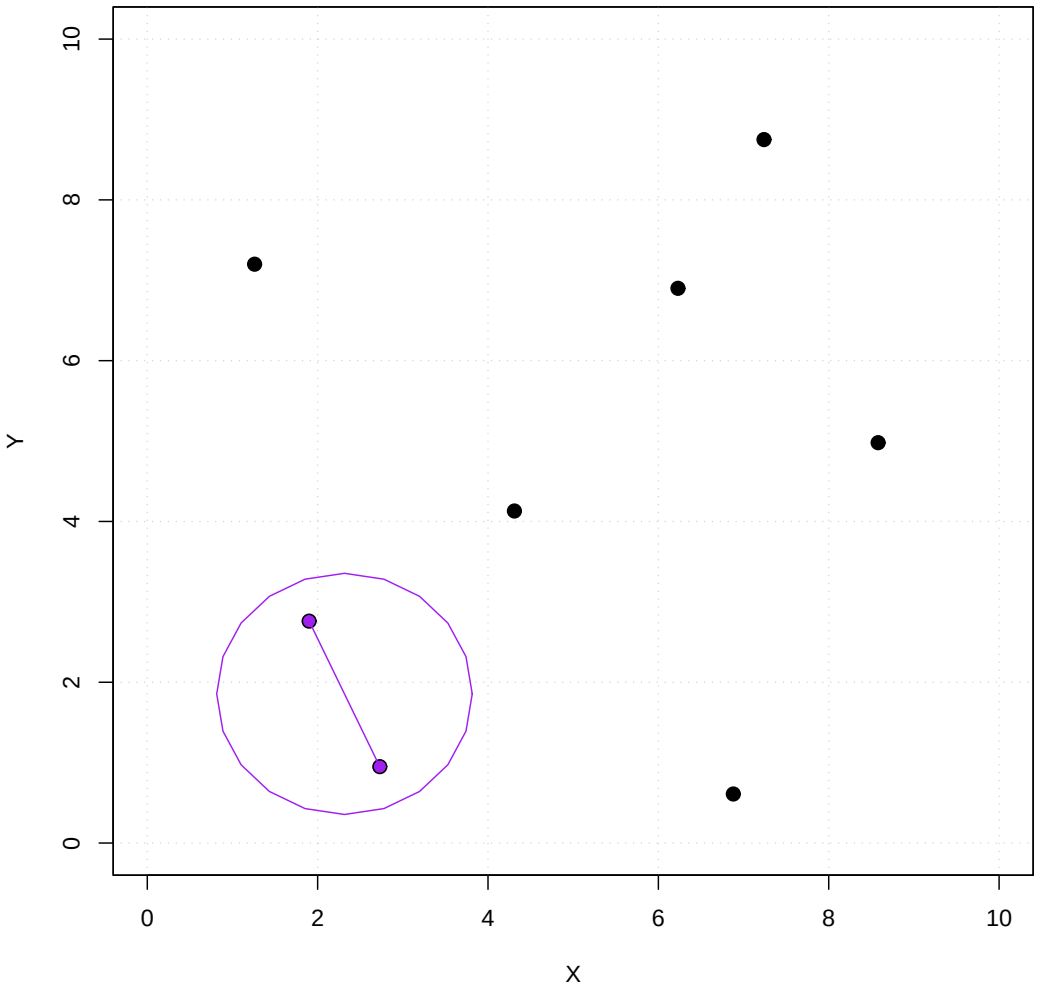

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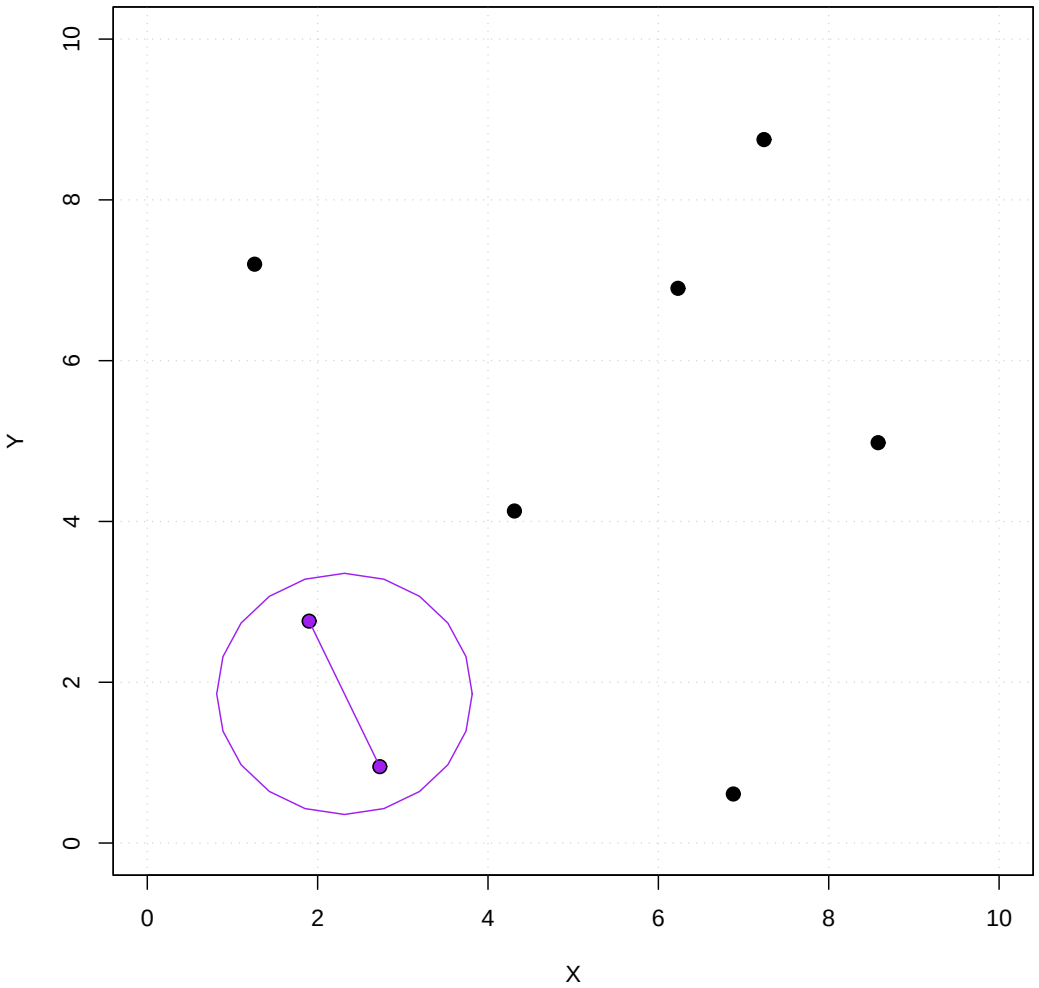

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- So $T(n) \in$ **?**

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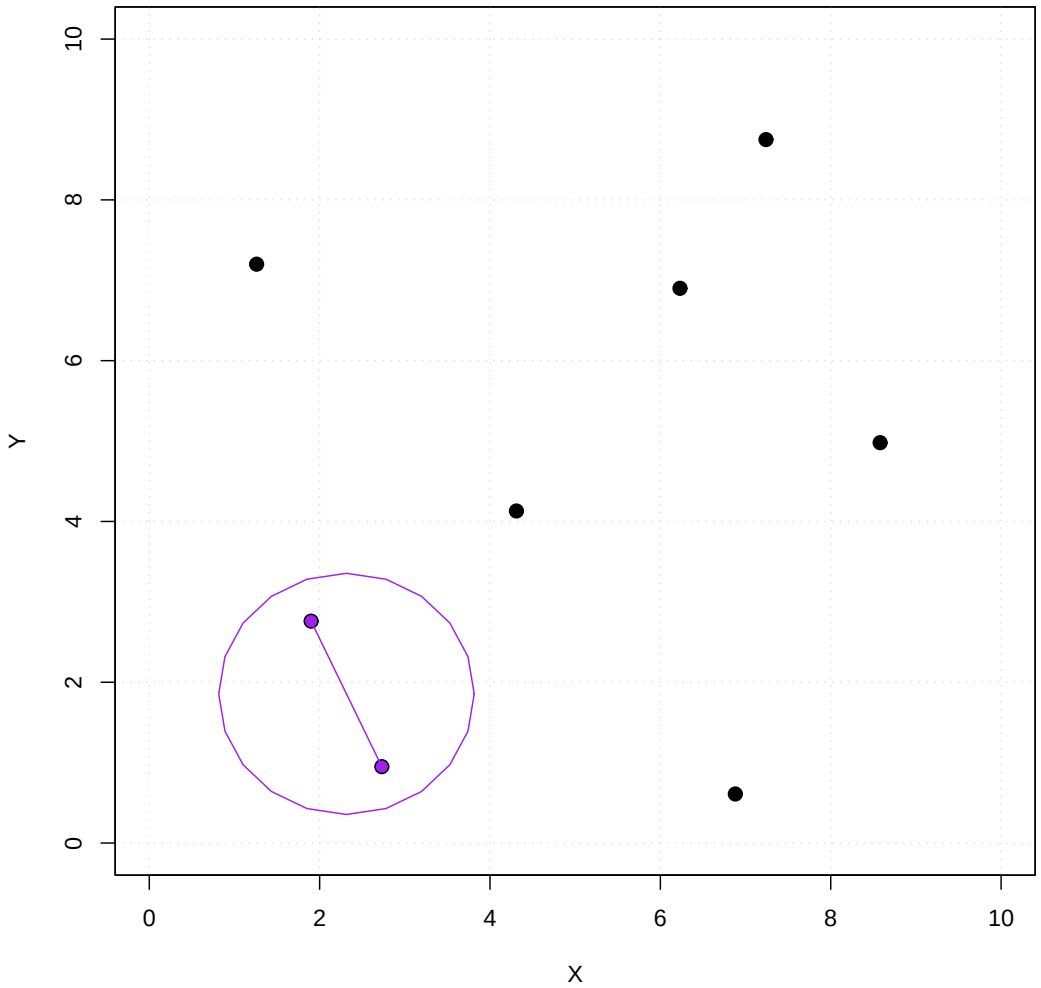

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QuickHull is similar, see book!

Assignments

- This week's assignments:
  - Section 4.1: Problems 5, 6, and 7
  - Section 4.2: Problems 1, 6, and 8
  - Section 4.3: Problems 1, 3, and 6
  - Section 4.4: Problems 2, 4, and 6
  - Section 4.5: Problems 2 and 6
  - Section 4.6: Problems 8 and 9

*Challenge problem