CS 483 - Data Structures and Algorithm Analysis

Lecture VI: Chapter 5, part 2; Chapter 6, part 1

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Outline

DFS & BFS Edge-Types

tree edge — Edge encountered by the search that leads to an as-yet unvisited node (DFS & BFS)

back edge — Edge leading to a previously visited vertex other than its immediate predecessor (DFS)

cross edge — Edge leading to a previously visited vertex other than its immediate predecessor (BFS)

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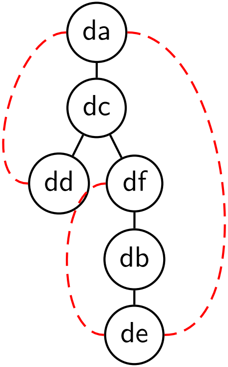

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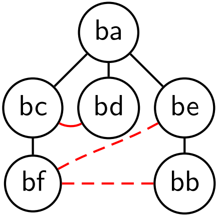

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Directed Graphs: A Review

- A *directed graph* (digraph) is a graph with *directed edges*
- We can use the same representational constructs: adjacency matrices & adjacency lists
- But there are some differences from the undirected case:
  - Adjacency matrix need not be symmetric
  - An edge in the digraph has only one node in an adjacency list
- We can still use DFS & BFS to traverse such graphs, but the resulting search forest is often more complicated
- There are now four edge types

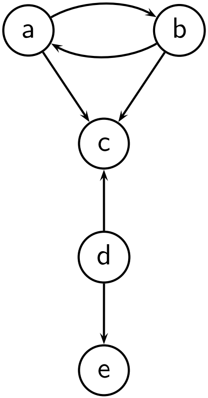
tree edge — Edge leading to an as-yet unvisited node

back edge — Edge leading from some vertex to a previously visited ancestor

forward edge — Edge leading from a previously visited ancestor to some vertex

cross edge — Remaining edge types

Directed Graphs: More Review



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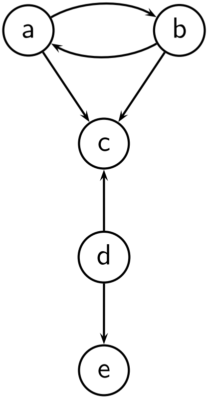

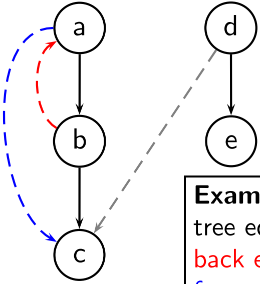

Example DFS forest:
tree edges
back edge
forward edge
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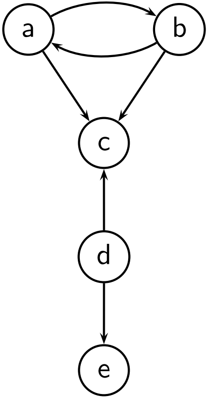
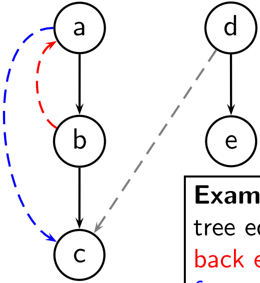
Directed Graphs: More Review





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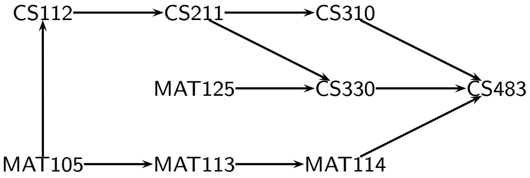
NOTE: A digraph with no back edges has no directed cycles. We call this a *directed acyclic graph* (DAG).

Representing Dependencies with DAGs

- Many real-world situations can be modeled with DAGs
- Consider problems involving dependencies (e.g., course pre-requisites):



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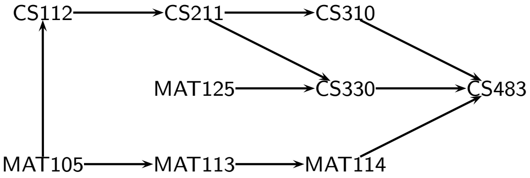

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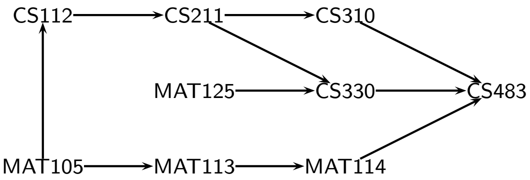

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- More generally: Order the vertices of a DAG such that for every edge, the vertex where the edge starts precedes the vertex where the edge ends?

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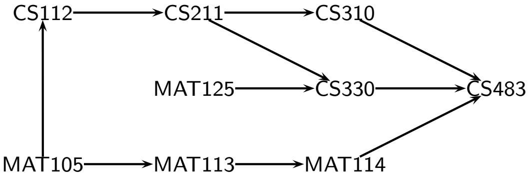

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- More generally: Order the vertices of a DAG such that for every edge, the vertex where the edge starts precedes the vertex where the edge ends?
- This problem is called *topological sorting*

Two Algorithms to Sort Topologies

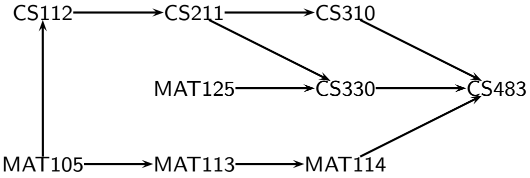
Algorithm 1

- Apply Depth-First Search
- Note the order in which the nodes become "*dead*" (popped off the traversal stack)
- Reverse the order; that is your answer
- Why does this work? When vertex v is popped off the stack, no vertex u with an edge (u, v) can be among the vertices popped of before v (otherwise (u, v) would be a back edge).

Algorithm 2

- Identify a *source* in the digraph (node with no in-coming edges)
- Break ties arbitrarily
- Record then delete the node, along with all edges from that node
- Repeat the process on the remaining subgraph
- When the graph is empty, you are done

Example On Algorithm 2



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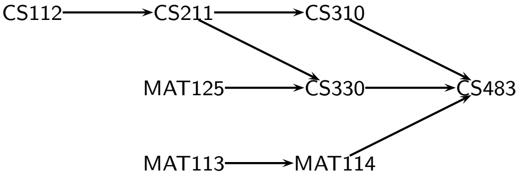

- MAT105

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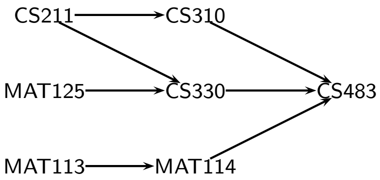

- MAT105
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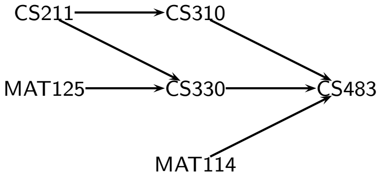

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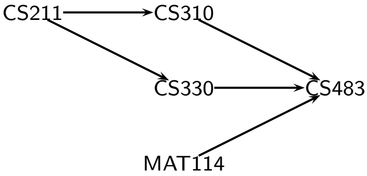

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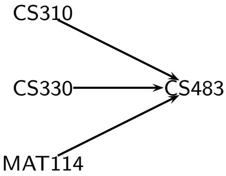

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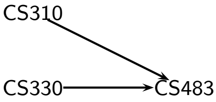

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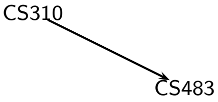

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Example On Algorithm 2

CS483

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Reviewing Combinations & Permutations

What is the difference between a *combination* and a *permutation*?

combination —

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combination — The number of ways of picking k *unordered* outcomes from n possibilities. We often write it as "*n choose k*". $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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combination — The number of ways of picking k *unordered* outcomes from n possibilities. We often write it as "*n choose k*". $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

permutation — A permutation is a rearrangement of the elements of an ordered list $\mathcal{S}$ into a one-to-one correspondence with $\mathcal{S}$ itself. ${}_nP_k = \frac{n!}{(n-k)!}$

Generating a Single Random Permutation

PermuteSet($A[0 \ldots n-1]$)

```
for i ⟵ 0 to n − 2
   j ⟵ RandInt(i + 1, n − 1)
   Swap(A, i, j)
```

- Basic operation is RandInt
- The loop is $\Theta(n)$

- In general, items in a permutation can be anything
- We think of them as ordered *sets* $\{a_0, a_2, \ldots, a_{n-1}\}$
- But we'll talk about them as a lists of integers for simplicity

For example:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

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Generating All Permutations

- Suppose we want to generate all permutations between 1 and n
- We can use Decrease-and-Conquer:
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Insert 3	123	132	312	321	231	213
		right to left			left to right	

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Satisfies *minimal-change*: each permutation can be obtained from its predecessor by exchanging just two elements

Generating n^{th} Permutation

- We can get the same ordering of permutations of n elements without generating the smaller permutations
  - We associate a direction with each element in the permutation: $\overrightarrow{3}\ \overleftarrow{2}\ \overrightarrow{4}\ \overleftarrow{1}$
  - A *mobile* component is one in which the arrow points to a smaller adjacent value (3 & 4 above, but not 1 & 2)

JohnsonTrotter(n)

Initialize the first permutation with $\overleftarrow{1}\ \overleftarrow{2} \cdots \overleftarrow{n}$
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Generating Subsets

- Given some universal set: $U = \{a_1, a_2, \cdots, a_n\}$, generate all possible subsets
- The set of all subsets is called a *power set*; there are 2^n of them

n	**subsets**							
0	$\emptyset$							
1	$\emptyset$	$\{a_1\}$						
2	$\emptyset$	$\{a_1\}$	$\{a_2\}$		$\{a_1, a_2\}$			
3	$\emptyset$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_2, a_3\}$	$\{a_1, a_2, a_3\}$

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- Can represent a set as a binary string:

000	001	010	011	100	101	
$\emptyset$	$\{a_3\}$	$\{a_2\}$	$\{a_2, a_3\}$	$\{a_1\}$	$\{a_1, a_3\}$	...

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- Is there an equivalent to minimal-change algorithm here?
  Yes: 000 001 011 010 110 111 101 100 (Gray code)

Fake-Coin Problem

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- You are provided a balance scale to compare sets of coins
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4. Otherwise examine the lighter pile in the same way

$$W(n) = W(\lfloor n/2 \rfloor) + 1 \text{ for } n > 1,\ W(1) = 0 \qquad \in \Theta(\lg n)$$

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But wait! This is not the most efficient way. What if you divided into *three* equal piles?

Multiplication á la Russe

- We want to compute the product of n and m, two positive integers
- But we only know how to add and multiple & divide by two
- If n is even, we can re-write: $n \cdot m = \frac{n}{2} \cdot 2m$
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n	m	
50	65	
25	130	130
12	260	
6	520	
3	1040	1040
1	2080	2080
		3,250

Median and Selection

- The *selection problem*: Find the k^{th} smallest element in a list of n numbers (the k^{th} order statistics)
- Finding the *median* is a special case: $k = \lceil n/2 \rceil$
- Brute force: Sort, then select the k^{th} value in the list: $O(n \lg n)$
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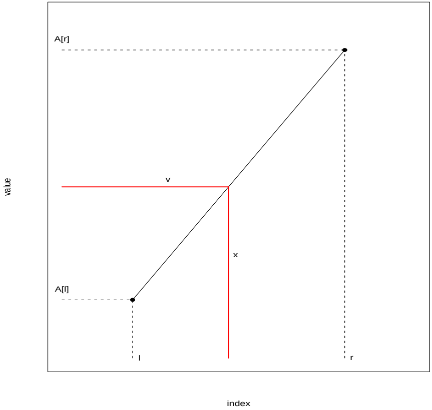
$$\boxed{\begin{array}{ccc} a_1 \cdots a_s & p & a_{s+1} \cdots a_n \\ \leq p & & \geq p \end{array}}$$

  - If $s = k$ then p solves the problem
  - If $s > k$ then the k^{th} smallest element in whole list is the k^{th} smallest element left-side sublist
  - If $s < k$ then the k^{th} smallest element in whole list is the $(k-s)^{th}$ smallest element right-side sublist
  - Average: $C(n) = C(n/2) + (n-1) \qquad \in \Theta(n)$

Interpolation Search

- Like BinarySearch, but more like a telephone book
- Rather than split the list in half, we interpolate the position based on the key value

  - $v :=$ search key value
  - $l :=$ left index
  - $r :=$ right index
  - $y = mx + b \Longrightarrow x = \frac{y-b}{m}$
  - $x = l + \left\lfloor \frac{(v-A[l])(r-l)}{A[r]-A[l]} \right\rfloor$
  - Average case: $O(\lg \lg n)$



Introduction to Transform & Conquer

- In many cases, one can transform a problem instance and solve the transformed problem
- Three variations of this idea are as follows:

instance simplicfication — Transform problem instance into a simpler or more convenient istance

representation change — Transform problem instance representations

problem reduction — Transform problem instance into an instance of different problem

Presorting

- Many questions about lists can be answered more easily when the list is already sorted
- The cost of the sort itself should be warranted

- Example: Element uniqueness

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- Example: Element uniqueness
  - Brute force: compare every element against every other element, $\Theta(n^2)$
  - Presort & scan: $T(n) = T_{\text{sort}}(n) + T_{\text{scan}} = \Theta(n \lg n) + \Theta(n) \in \Theta(n \lg n)$
- Example: Search

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- Example: Element uniqueness
  - Brute force: compare every element against every other element, $\Theta(n^2)$
  - Presort & scan: $T(n) = T_{\text{sort}}(n) + T_{\text{scan}} = \Theta(n \lg n) + \Theta(n) \in \Theta(n \lg n)$

- Example: Search
  - Brute force: linear search: $\Theta(n)$
  - Presort & binary search: $\Theta(n \lg n) + \Theta(\lg n) \in \Theta(n \lg n)$
  - Presorting does not help with one search, though perhaps it will with *many* searches on the same list

- Example: Computing a *mode* (most frequent value)

Presorting

- Many questions about lists can be answered more easily when the list is already sorted
- The cost of the sort itself should be warranted

- Example: Element uniqueness
  - Brute force: compare every element against every other element, $\Theta(n^2)$
  - Presort & scan: $T(n) = T_{\text{sort}}(n) + T_{\text{scan}} = \Theta(n \lg n) + \Theta(n) \in \Theta(n \lg n)$

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  - Presorting does not help with one search, though perhaps it will with *many* searches on the same list

- Example: Computing a *mode* (most frequent value)
  - Brute force: scan list an store count in auxiliary list then scan auxiliary list for highest frequency, worst case time $\Theta(n^2)$
  - Presort, Longest-run: sort then scan through list looking for the longest run of a value, $\Theta(n \lg n)$

Gaussian Elimination

- Problem: Solve a system of n linear equations with n unknowns

$$\begin{array}{ccc}
a_{11}x_1 + a_{12}x_2 + & \cdots & + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + & \cdots & + a_{2n}x_n = b_2 \\
 & \vdots & \\
a_{n1}x_1 + a_{n2}x_2 + & \cdots & + a_{nn}x_n = b_n
\end{array}$$

- This can be written as $A\vec{x} = \vec{b}$
- Gaussian Elimination first asks us to transform the problem to a different one, one that has the same solution: $A'\vec{x} = \vec{b}'$
- The transformation yields a matrix with all zeros below its main diagonal:

$$A' = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \ddots & & \\ 0 & 0 & \cdots & a'_{nn} \end{bmatrix}, \quad \vec{b}' = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ v'_n \end{bmatrix}$$

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A simple backward substitution method can be used to obtain the solution now!

Gaussian Elimination: Obtaining An Upper-Triangle Coefficient Matrix

- Solutions to the system are invariant to three *elementary operations*:
  - Exchange two equations of the system
  - Replace an equation with its nonzero multiple
  - Replace an equation with a sum or difference of this equation and some multiple of another
- Consider the following example:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 4x_1 + x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} \text{row } 2 \leftarrow row2 - row1 * \frac{4}{2} \\ \text{row3} \leftarrow row3 - row1 * \frac{1}{2} \end{array}$$

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Gaussian Elimination: Obtaining An Upper-Triangle Coefficient Matrix

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Upper-triangle form!

LU Decomposition

- If we track the row multiples used during Gaussian elimination, we can construct a lower-triagonal matrix (with one's on the diagonal)
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$
- We can also consider the upper-triangular matrix produced by Gaussian elimination, leaving off the $\vec{b}'$ vector:
$$U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$
- It turns out that $A = LU$, so we can re-write our original system as $LU\vec{x} = \vec{b}$
- We can split this into two steps, and solve each with back substitution: $L\vec{y} = \vec{b}$ then $U\vec{x} = \vec{y}$
- Advantage: We can solve many systems with different $\vec{b}$ vectors in the same way, with minimal additional effort

LU Decomposition

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NOTE: We can actually store L and U in the *same matrix* to save space.

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- Advantage: We can solve many systems with different $\vec{b}$ vectors in the same way, with minimal additional effort

Matrix Inversion

- The inverse of a matrix, denoted A', is defined as $AA' = I$, where I is the identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- But this can be written as a series of systems of linear equations, $A\vec{x}^j = \vec{e}^j$ where:

$$A' = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \ddots & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix}$$

  - $\vec{x}^j$ is the j^{th} column of the inverse matrix
  - $\vec{e}^j$ is the j^{th} column of the identity matrix
- We can compute the LU decomposition of A, then systematically attempt to solve for each column of the inverse
- If compute a U with zeros on the diagonal, there is no inverse and A is said to be *singular*

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$$I = \begin{array}{c} \vec{e}^1 \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \end{array}$$

$$A' = \begin{array}{c} \vec{x}^1 \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \ddots & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix} \end{array}$$

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Matrix Inversion

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$$I = \begin{array}{cc} & \vec{e}^2 \\ & \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{array}\right] \end{array}$$

- But this can be written as a series of systems of linear equations, $A\vec{x}^j = \vec{e}^j$ where:
  - $\vec{x}^j$ is the j^{th} column of the inverse matrix
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$$A' = \begin{array}{c} \vec{x}^2 \\ \left[\begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \ddots & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right] \end{array}$$

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Book Topics Skipped in Lecture

- In section 5.5:
  - *Josephus Problem* (pp. 182–184)
- In section 5.6:
  - *Search and Insertion in a Binary Search Tree* (pp. 188–189)
- In section 6.2:
  - The GAUSSELIMNATION and BETTERGAUSSELIMINATION algorithms in detail (pp. 202–203)
  - *Computing a Determinant* (pp. 206–207)

Assignments

- This week's assignments:
  - Section 5.3: Problems 1, 2, & 5
  - Section 5.4: Problems 1, 2, & 5
  - Section 5.5: Problems 2 & 4
  - Section 5.6: Problems 2 & 6
  - Section 6.1: Problems 1, 5, & 6
  - Section 6.2: Problems 1, 2, & 7

Project II: Balanced Trees

See project description at:
`http://www.cs.gmu.edu/~pwiegand/cs483/assignments.htm`

The project will be **due by midnight April 7**.