Nash's Theorem

Every game with a finite number of actions for each player where each player's utilities are consistent with the axioms of utility theory has an equilibrium in mixed strategies.

Idea 1: Reaction correspondences. Player i's reaction correspondence r_i maps each strategy profile σ to the set of mixed strategies that maximize player i's payoff when her opponents play σ_{-i}. Note that r_i depends only on σ_{-i}, so we don't really need all of σ, but it will be useful to think of it this way. Let r be the Cartesian product of all r_i. A fixed point of r is a σ such that $\sigma \in r(\sigma)$, so that for each player, $\sigma_i \in r_i(\sigma)$. Thus a fixed point of r is a Nash equilibrium.

Kakutani's FP theorem says that the following are sufficient conditions for $r : \Sigma \to \Sigma$ to have a FP.

1. Σ is a compact, convex, nonempty subset of a finite-dimensional Euclidean space.

   Satisfied, because it's a simplex

2. $r(\sigma)$ is nonempty for all σ

   Each player's payoffs are linear, and therefore continuous, in her own mixed strategy. Continuous functions on compact sets attain maxima.

3. $r(\sigma)$ is convex for all σ

   Suppose not. Then $\exists \sigma', \sigma''$ such that $\lambda\sigma' + (1-\lambda)\sigma'' \notin r(\sigma)$ But for each player i,

   $$u_i(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}) =$$

   $$\lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda) u_i(\sigma_i'', \sigma_{-i})$$

   so that if both σ' and σ'' are best responses to σ_{-i}, then so is their weighted average.

4. $r(\cdot)$ has a closed graph

   The correspondence $r(\cdot)$ has a closed graph if the graph of $r(\cdot)$ is a closed set. Whenever the sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$, with $\hat{\sigma}^n \in r(\sigma^n) \forall n$, then $\hat{\sigma} \in r(\sigma)$ (same as upper hemicontinuity)

   Suppose that there is a sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ such that $\hat{\sigma}^n \in r(\sigma^n)$for every n, but $\hat{\sigma} \notin r(\sigma)$. Then there exists $\epsilon > 0$ and σ' such that

   $$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$$

   Then, for sufficiently large n,

   $$u_i(\sigma_i', \sigma_{-i}^n) > u_i(\sigma_i', \sigma_{-i}) - \epsilon > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon$$

   $$> u_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \epsilon$$

   which means that σ_i' does strictly better against σ_{-i}^n than $\hat{\sigma}_i^n$ does, contradicting our assumption.

An Auction Game

Suppose I run a first price auction for a painting. There are two bidders, and it is common knowledge that $v_i \sim U[0,1]$ for both. How much should the seller expect to make from this auction?

Well, let's solve the game. Suppose I am a bidder with valuation v. My strategy s is a mapping from v to b, my bid. Let's make two assumptions:
(1) $s(\cdot)$ is strictly increasing and differentiable (this is restrictive).
(2) $s(v) \le v \quad \forall v$ (this is rational, and also implies $s(0) = 0$.)

We'll restrict attention to the case where both participants use the same s. This makes sense because they are *a priori* identical.

Then the bidder with the higher valuation wins.
Therefore Pr(I win | I have value v_i) $= v_i$.
If I win, my payoff is $v_i - s(v_i)$.
Therefore, my expected payoff is $v_i(v_i - s(v_i)) = g(v_i)$.

How can we analyze deviations to an arbitrary strategy $s'(\cdot)$ satisfying the two conditions above? It doesn't make sense to bid below 0 or above 1, and $s'(\cdot)$ is continuous, increasing, and differentiable. Therefore, we can *simulate* s' by submitting a fake valuation to s.
Then, the non-deviation condition becomes:

$$v_i(v_i - s(v_i)) \geq v(v_i - s(v)) \forall \text{fake values } v$$

Proposition: $s(v) = v/2$ satisfies this. Why?
LHS is $v_i^2/2$. RHS is $vv_i - v^2/2$. So we need

$$\frac{v_i^2}{2} - vv_i + \frac{v^2}{2} \geq 0$$

$$\Rightarrow (1/2)(v - v_i)^2 \geq 0$$

which is true.

[Note: not a dominant strategy, only equilibrium]

How do we actually find the solution? In this case through a differential equation:
In order for $s(\cdot)$ to satisfy $v_i(v_i - s(v_i)) \geq v(v_i - s(v))$, $g(v) = v(v_i - s(v))$ must be maximized at $v = v_i$. Therefore $g'(v_i) = 0$.

$$g'(v) = v_i - s(v) - vs'(v)$$

$$\text{Therefore } s'(v_i) = 1 - \frac{s(v_i)}{v_i}$$

This is solved by $s(v_i) = v_i/2$.

So, what can the auctioneer expect to make? Second order statistic of the uniform distribution with 2 samples is 2/3, therefore 1/3.

Second Price Auctions

Highest bidder wins, but pays the amount of the second highest bid.

Dominant strategy to bid true valuation. Why?

Consider alternate bid $b_i + \delta$. Raised bid affects outcome only if highest other bid b_j is in-between b_i and $b_i + \delta$. But then you end up paying more than you value the item for!

Consider alternate bid $b_i - \delta$. Lowered bid affects outcome only if highest other bid b_k is in-between b_i and $b_i - \delta$. But then you lose and get 0 when you could have won with a non-negative payoff!

Expected revenue in the uniform setting? $n-$1st order statistic of n draws, so $\frac{n-1}{n+1}$. Exactly the same! An example of the *revenue equivalence theorem*: very sketchily, in auctions where the bidder with the highest valuation wins, all bidders are risk neutral, and a couple of other conditions, the seller's expected revenue is the same.