The Auction Game With n Players

What is the new probability of winning? v_i^{n-1}.

So what is bidder i's expected payoff?

$$G(v_i) = v_i^{n-1}(v_i - s(v_i))$$

Expected payoff of a fake v?

$$G(v) = v^{n-1}(v_i - s(v))$$

Therefore, it must be that

$$v_i^{n-1}(v_i - s(v_i)) \geq v^{n-1}(v_i - s(v)) \quad \forall v \in [0,1]$$

So $G(v)$ must be maximized at $v = v_i$, or $G'(v_i) = 0$.

$$\begin{aligned} G'(v) &= (n-1)v^{n-2}(v_i - s(v)) - v^{n-1}s'(v) \\ &= -v^{n-1}s'(v) + (n-1)v^{n-2}v_i - (n-1)v^{n-2}s(v_i) \end{aligned}$$

Dividing by $(n-1)v^{n-2}$ and setting to 0, we get:

$$v_i - s(v_i) - \frac{v_i s'(v_i)}{n-1} = 0$$

$$\Rightarrow s'(v_i) = (n-1)(1 - \frac{s(v_i)}{v_i})$$

This is solved by $s(v_i) = (\frac{n-1}{n})v_i$.

Key note: have to be more aggressive as n grows!

Seller revenue? For the standard uniform distribution, the expectation of $\frac{n-1}{n}\times$ the nth order statistic from n draws, which is $\frac{n}{n+1}$, so $\frac{n-1}{n+1}$.

Note that this is the same as the expected revenue of the second price auction with n bidders with the same valuation distribution!

Reserve Prices

Suppose the seller can set a (secret) *reserve price* r, below which she will not sell the item. In a second price auction, the winner now pays the max of the second highest bid and r. This is still a truthful mechanism for bidders.

What about the seller? Suppose I value it at u, should I set the reserve price at u?

Not necessarily! Consider the case of a single bidder with $v \sim U[0,1]$. Seller wishes to maximize $\Pr(v > r)\dot{r} = r(1-r)$ which is maximized at $r = 1/2$!

Common Value Auctions and the Winner's Curse

Suppose I'm selling an item, and everyone has a noisy signal of the value of that item (for example, on Storage Hunters / Auctions Hunters in reality TV, where folks bid on the contents of an abandoned self-storage locker).

If everyone bids their signal, in expectation the winner will be the one with the highest signal, which will be significantly more than then expected value! This is the winner's curse.

The equilibrium of this game would be to take that into account when bidding.

$$\text{Bid } = \mathbb{E}(v(s_i)|s_i, \text{all } s_j \leq s_i \text{ for } j \neq i)$$

Mechanism Design

In some senses this is *inverse game theory*. How do you set up the rules of the game in order to induce desirable behavior among the agents playing the game? Ideally the implementation should be in as strong a solution concept as possible – dominant strategies are preferred to Bayes-Nash equilibria.

Let A be a set of alternatives which are the outcomes to be decided between. Let $v_i : A \to \mathbb{R}$ and $v_i \in V_i \subseteq \mathbb{R}^A$ be a commonly known set of possible valuation functions. Notationally, $v = (v_1, \dots, v_n) = (v_i, v_{-i})$.

A *direct revelation mechanism* is:
(1) a social choice function $f : v_1 \times \dots \times v_n \to A$
(2) a vector $(p_1, \dots p_n)$ of payment functions, where
$p_i : v_1 \times \dots \times v_n \to \mathbb{R}$ is the amount i pays.

(The direct revelation part means that the only actions available to agents are to in some way report preferences to the mechanism)

The classic desideratum: *incentive compatibility* (agents report truthful information about their preferences due to their own self-interest): $\forall i, \forall v_1 \in V_1, \dots, v_n \in V_n$, and every $v_i' \in V_i$, if $a = f(v_i, v_{-i}), a' = f(v_i', v_{-i})$, then

$$v_i(a) - p(v_i, v_{-i}) \geq v_i(a') - p(v_i', v_{-i})$$

Vickrey-Clarke-Groves Mechanisms

A mechanism is VCG if:
(1) $f(v_1, \dots, v_n) \in \text{argmax}_{a \in A} \sum_i v_i(a)$
(2) For some functions $h_1, \dots h_n$ where $h_i : V_{-i} \to \mathbb{R}$ (not dependent on v_i) we have $\forall v_1 \in V_1, \dots v_n \in V_n$:

$$p_i(v_1, \dots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \dots, v_n))$$

The first condition says that the selected outcome must be the one that maximizes social welfare. The second one says that each player is **paid** an amount equal to the sum of the others' valuations. h_i has no strategic implications, since it only depends on others' actions. This aligns all players incentives with the goal of maximizing social welfare.

Theorem: Every VCG mechanism is incentive compatible.

Proof: Consider i, v_{-i}, v_i, and v_i'. We need to show that the utility for player i when declaring $v_i \geq$ the utility when declaring v_i'.
Let $a = f(v_i, v_{-i}), a' = f(v_i', v_{-i})$ (social choices).
Utility when declaring v_i? $v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i})$.
Utility when declaring v_i'? $v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i})$.
But we know that $a = f(v_i, v_{-i})$ maximizes social welfare across all alternatives. Therefore

$$v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$$