## Optimal Stopping

(Most of this lecture based on Chapter 2 of Ferguson's "Optimal Stopping").

Choose a time to take an action given a sequence of observed random variables.

Wish to maximize expected payoff or minimize expected cost

Three (finite-horizon) examples: the Cayley-Moser Problem, the Secretary Problem, and the Parking Problem.



## The Cayley-Moser Problem

Buying a house, selling an asset, or searching for a job.

$m$ objects, with i.i.d. values $X_1, X_2, \ldots, X_n$ from a known distribution.

At each time $i$, you get to observe $X_i$ and then make a "take it or leave it" decision. If you take it, you get $X_i$ as your reward and the process is over. If you leave it, search continues.

You will look at least at the first option. If you reach the $n$th option you will choose that one.

Let's suppose $X_i \sim U[0,1]$.



## Solving the Problem

What if $m = 2$? When would you take $X_1$? When $X_1 > 0.5$.

Can we generalize this? What's the value of not choosing $X_j$ and continuing the search? Would we rather do that or choose $X_j$ and stop?

$V_j = \max\{X_j, \mathbb{E}(V_{j+1})\}$

Note that the dependence is entirely on the number of stages left to go. So define $A_{n-j} = \mathbb{E}(V_{j+1})$. Then:

$$
\begin{aligned}
A_0 &= -\infty \\
A_1 &= \mathbb{E}[X_1] \\
A_{j+1} &= \mathbb{E}\max\{X, A_j\} \\
&= \int_{-\infty}^{A_j} A_j \, dF(x) + \int_{A_j}^{\infty} x \, dF(x)
\end{aligned}
$$



Specializing to the uniform $[0,1]$ distribution:

$$
\begin{aligned}
A_{j+1} &= \int_0^{A_j} A_j \, dx + \int_{A_j}^1 x \, dx \\
&= (A_j^2 + 1)/2
\end{aligned}
$$

Then $A_2 = 5/8, A_3 = 89/128, \ldots$.

## The Secretary Problem

One position available with $n$ applicants; the relative ranking is complete.

Applicants are interviewed sequentially in a random order, and you have to either hire the applicant or reject him immediately. There is no recall.

The only available information is on rank, not on actual values. Therefore, the decision can only be based on relative ranks of applicants interviewed so far.

Objective: select the best applicant. If you do so, you win. Otherwise you lose.

What do you think the probability of succeeding is, when using an optimal rule with large $n$?



## Solving the Secretary Problem

When does it make sense to accept an applicant? Only when he is best among those already observed (otherwise lose for sure). We call such applicants **candidates**.

When to make an offer to a candidate at stage $j$? What is the probability of winning with such a candidate? The same as the probability that the best of the first $j$ is the best overall: $j/n$.

Let $W_j$ be the probability of winning when using an optimal rule that does not accept any of the first $j$ applicants. Note $W_j \geq W_{j+1}$ because all rules available at $j+1$ are also available at $j$.

It is optimal to stop with a candidate at stage $j$ if $j/n \geq W_j$. Then it is also optimal to stop with a candidate at $j+1$ since $(j+1)/n > j/n \geq$



$W_j \geq W_{j+1}$. Therefore an optimal rule is of the form $N_r$: "Reject the first $r-1$ applicants and then accept the next candidate (relatively best applicant) if any."

What is the probability of winning using $N_r$?

$$
\begin{aligned}
P_r &= \sum_{k=r}^{n} \Pr(\text{Applicant } k \text{ is best and selected}) \\
&= \sum_{k=r}^{n} \Pr(\text{Applicant } k \text{ is best}) \Pr(k \text{ is selected} \mid \text{best}) \\
&= \sum_{k=r}^{n} \frac{1}{n} \Pr(\text{best of first } k-1 \text{ appears before stage } r) \\
&= \sum_{k=r}^{n} \frac{1}{n} \frac{r-1}{k-1} = \frac{r-1}{n} \sum_{k=r}^{n} \frac{1}{k-1}
\end{aligned}
$$

(where $\frac{r-1}{r-1}$ represents 1 when $r=1$; the third step is because each applicant is *a priori* equally likely to be best, and then we want to make sure that the best of the first $k-1$ does not appear at a time when we would pick him, that is from stage $r$ onwards).

Now, we want to choose $r$ so as to maximize $P_r$. Can do this explicitly for small $n$.

In the limit as $n \to \infty$, let $x$ be $r/n$ and $t$ be $k/n$. Then the above expression becomes $P(x) = x \int_x^1 \frac{1}{t}\, dt = -x \ln x$. Take the derivative and set to zero, and you find that the optimal rule is to use $n/e$ as the cutoff, and then the optimal applicant is selected with $\Pr(1/e)$!

## The Parking Problem

Driving along an infinite street (the only one in the world) to the theater.

Want to park as close to the theater as possible, and you're not allowed to turn around.

Assume the street is populated with parking spots at each integer point on the real line, and that the theater is located at $T > 0$. You are driving towards $T$ from the left. Each spot is occupied with probability $p$ (i.i.d. Bernoulli r.v.s)

You can't see spot $j+1$ when you are at $j$. Can't return to a previous spot. If you park at spot $j$, you lose $|T-j|$. If you reach $T$ without having parked you have to keep driving to the next open spot past it.



We can treat this as a finite horizon problem. If you reach $T$ your payoff is 0 if it is available, and $(1-p)+2p(1-p)+3p^2(1-p)+\ldots = 1/(1-p)$ otherwise.

It's obvious that if it is optimal to stop at $j$ then it is optimal to stop at $j+1$. So we can use a threshold rule $N_r$: continue until you are $r$ places from the destination and then park at the first available spot.

How do you compute $r$? Let $P_r$ denote the expected cost using the rule $N_r$. Then $P_0 = p/(1-p)$, and $P_r = (1-p)r + pP_{r-1}$. Can show by induction that

$$P_r = r + 1 + \frac{2p^{r+1}-1}{1-p}$$

Clearly true for $P_0$. Suppose it is true for $r-1$. Then

$$P_r = (1-p)r + pP_{r-1} = (1-p)r + pr + p(2p^r-1)/(1-p)$$
$$= r + 1 + \frac{2p^{r+1}-1}{1-p}$$

Now, $P_{r+1} - P_r = 1 - 2p^{r+1}$. This is increasing in $r$, so we want to find the first $r$ for which this difference is non-negative. So if $p \leq 1/2$, get to $T$ before looking. If $p = .9$, start looking 6 places before the destination.

## Variants of Interest

Costly sequential search: can still be infinite horizon, but pay a cost $c$ in order to sample the next opportunity.

Search with recall: can go back to previous opportunities, perhaps up to a few.

Selection of $k$ candidates.