Height Balance: AVL Trees

Definition:

An ***AVL tree*** is a binary search tree in which the heights of the left and right subtrees of the root differ by at most 1 and in which the left and right subtrees are again AVL trees.

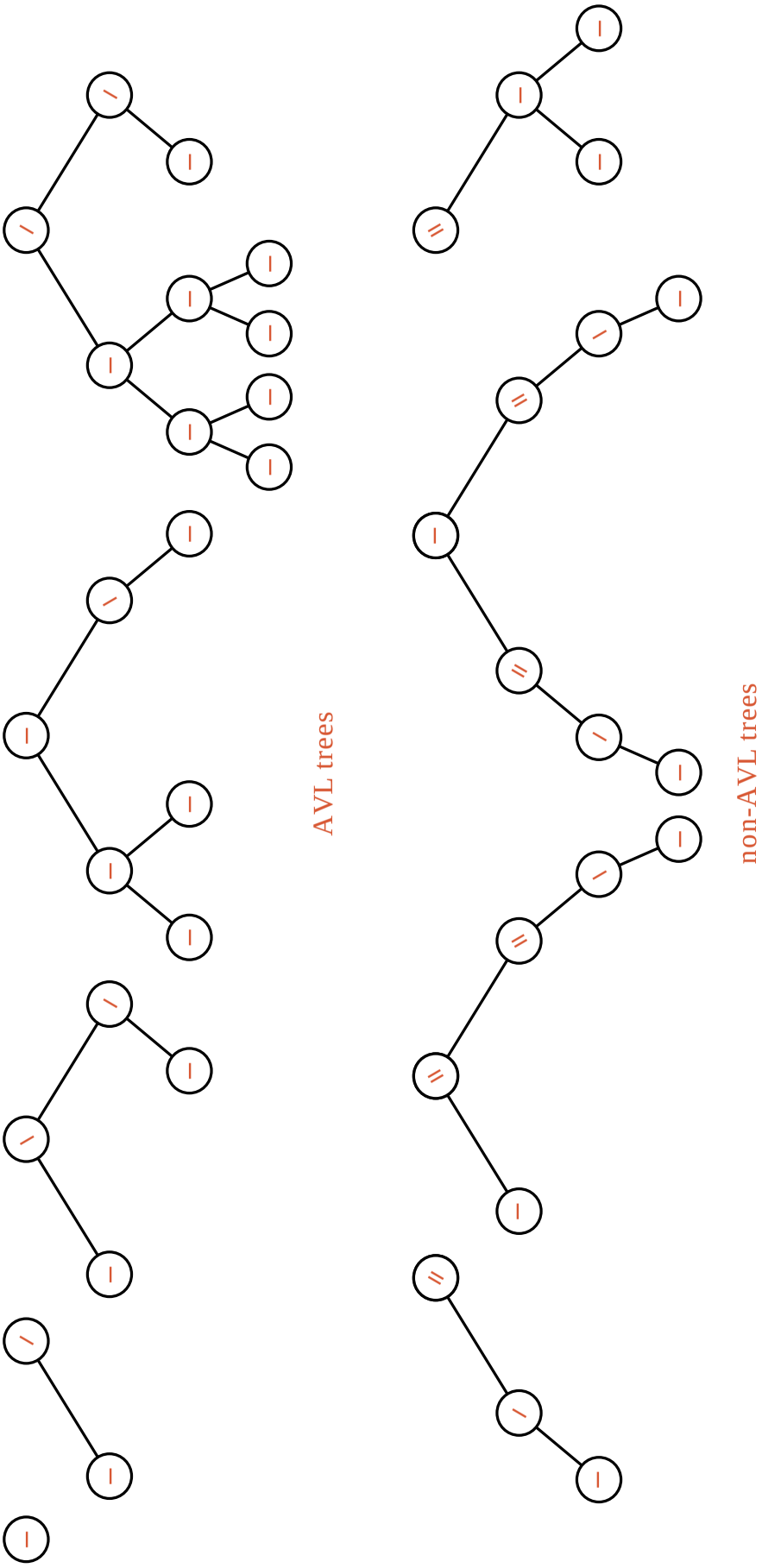
With each node of an AVL tree is associated a ***balance factor*** that is ***left higher***, ***equal***, or ***right higher*** according, respectively, as the left subtree has height greater than, equal to, or less than that of the right subtree.

History:

The name *AVL* comes from the discoverers of this method, G. M. Adel'son-Vel'skiĭ and E. M. Landis. The method dates from 1962.

Convention in diagrams:

In drawing diagrams, we shall show a left-higher node by '/,' a node whose balance factor is equal by '−,' and a right-higher node by '\.'



Examples of AVL trees and other binary trees
Transp. 38, Sect. 10.4, Height Balance: AVL Trees

C++ Conventions for AVL Trees

- We employ an enumerated data type to record balance factors:

```
enum Balance_factor { left_higher, equal_height, right_higher };
```

- AVL nodes are structures derived from binary search tree nodes with balance factors included:

```
template <class Record>
struct AVL_node: public Binary_node<Record> {
// additional data member:
   Balance_factor balance;
// constructors:
   AVL_node( );
   AVL_node(const Record &x);
// overridden virtual functions:
   void set_balance(Balance_factor b);
   Balance_factor get_balance( ) const;
};
```

- Methods for balance factors:

```
template <class Record>
void AVL_node<Record> :: set_balance(Balance_factor b)
{
   balance = b;
}

template <class Record>
Balance_factor AVL_node<Record> :: get_balance( ) const
{
   return balance;
}
```

Virtual Methods and Dummy Methods

A C++ compiler must reject a call such as left->get_balance( ), since left might point to a Binary_node that is not an AVL_node. We resolve this difficulty by including *dummy* versions of get_balance( ) and set_balance( ) in the underlying Binary_node structure.

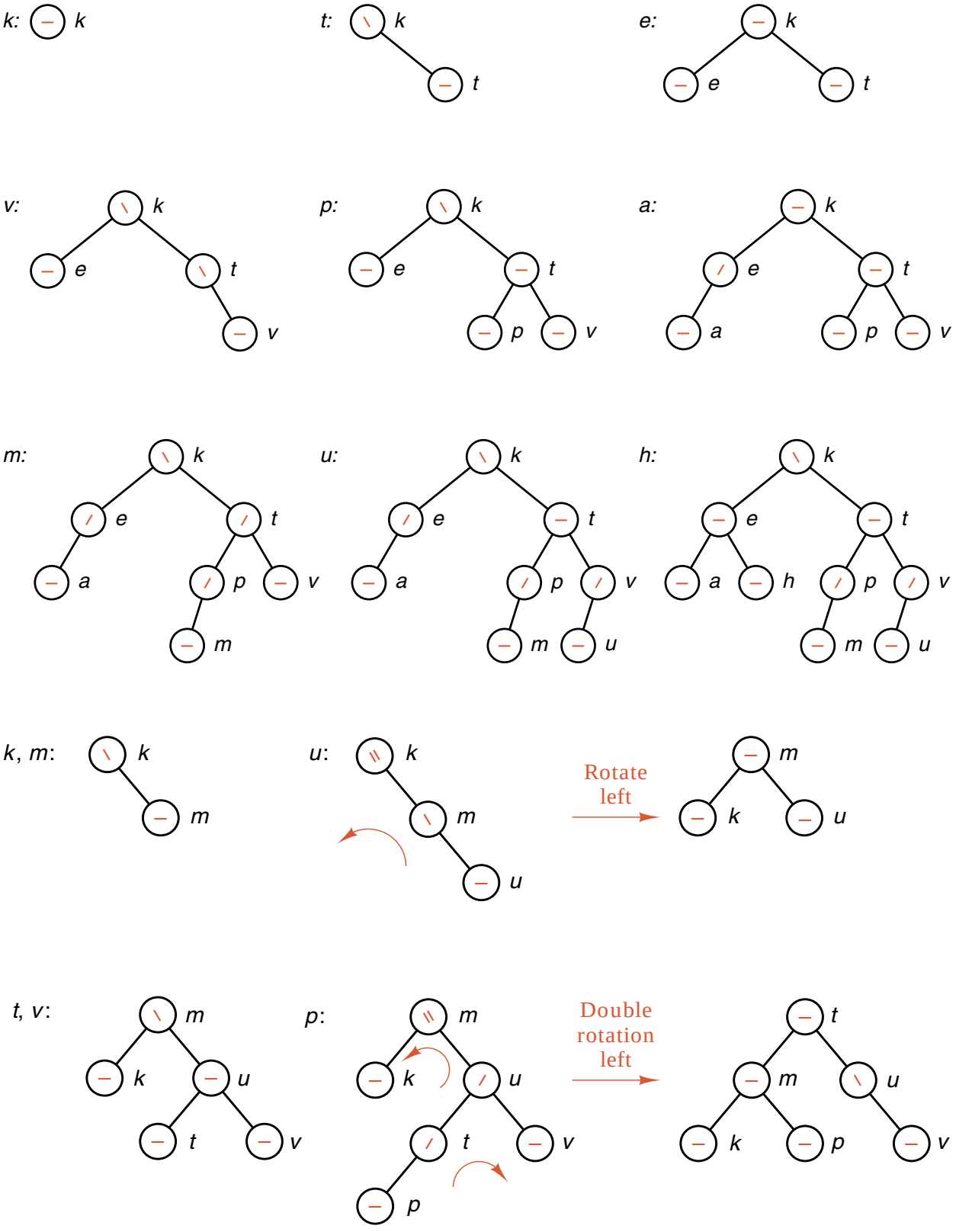
The correct choice between the AVL version and the dummy version of the method can only be made at run time, when the type of the object *left is known. To allow this, we must declare Binary_node versions of set_balance and get_balance as **virtual** methods; since these are dummies, they need do nothing.

```
template <class Entry>
struct Binary_node {
//    data members:
   Entry data;
   Binary_node<Entry> *left;
   Binary_node<Entry> *right;
//    constructors:
   Binary_node( );
   Binary_node(const Entry &x);
//    virtual methods:
   virtual void set_balance(Balance_factor b);
   virtual Balance_factor get_balance( ) const;
};
```

Class declaration for AVL trees:

```
template <class Record>
class AVL_tree: public Search_tree<Record> {
public:
   Error_code insert(const Record &new_data);
   Error_code remove(const Record &old_data);
private:                              //    Add auxiliary function prototypes here.
};
```

Insertions into an AVL tree



Public Insertion Method

```
template <class Record>
Error_code AVL_tree<Record> :: insert(const Record &new_data)
/* Post:  If the key of new_data is already in the AVL_tree, a code of duplicate_error
          is returned.  Otherwise, a code of success is returned and the Record
          new_data is inserted into the tree in such a way that the properties of an AVL
          tree are preserved.
   Uses:  avl_insert.  */
{
   bool taller;                          //    Has the tree grown in height?
   return avl_insert(root, new_data, taller);
}
```

Recursive Function Specifications

```
template <class Record>
Error_code AVL_tree<Record> ::
              avl_insert(Binary_node<Record> * &sub_root,
                         const Record &new_data, bool &taller)
/* Pre:   sub_root is either NULL or points to a subtree of the AVL_tree
   Post:  If the key of new_data is already in the subtree, a code of duplicate_error
          is returned.  Otherwise, a code of success is returned and the Record
          new_data is inserted into the subtree in such a way that the properties of
          an AVL tree have been preserved.  If the subtree is increased in height, the
          parameter taller is set to true; otherwise it is set to false.
   Uses:  Methods of struct AVL_node; functions avl_insert recursively,
          left_balance, and right_balance.  */
```

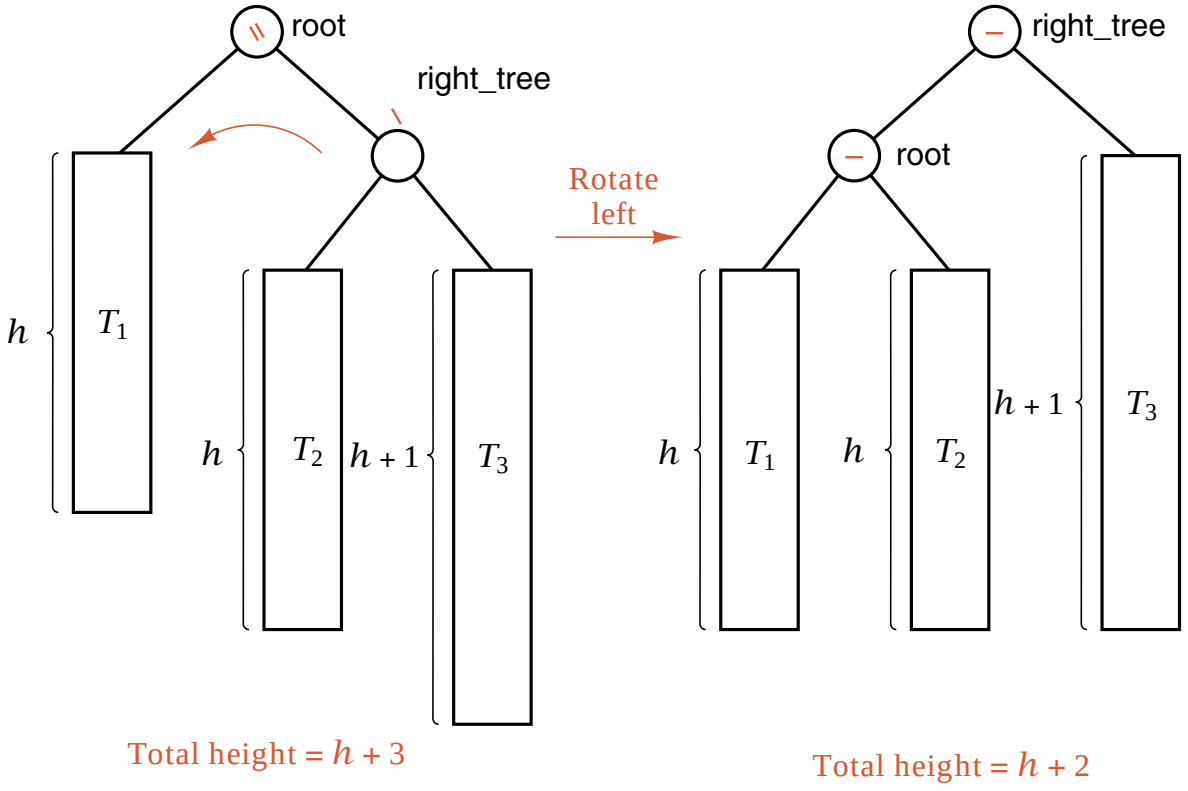
Recursive Insertion

```
{
   Error_code result = success;
   if (sub_root ==  NULL) {
      sub_root = new AVL_node<Record>(new_data);
      taller = true;
   }

   else if (new_data ==  sub_root->data) {
      result = duplicate_error;
      taller = false;
   }

   else if (new_data < sub_root->data) {  //     Insert in left subtree.
      result = avl_insert(sub_root->left, new_data, taller);
      if (taller ==  true)
         switch (sub_root->get_balance( )) {  //     Change balance factors.
         case left_higher:
            left_balance(sub_root);
            taller = false;                 //     Rebalancing always shortens the tree.
            break;

         case equal_height:
            sub_root->set_balance(left_higher);
            break;

         case right_higher:
            sub_root->set_balance(equal_height);
            taller = false;
            break;
         }
   }
```

Recursive Insertion, Continued

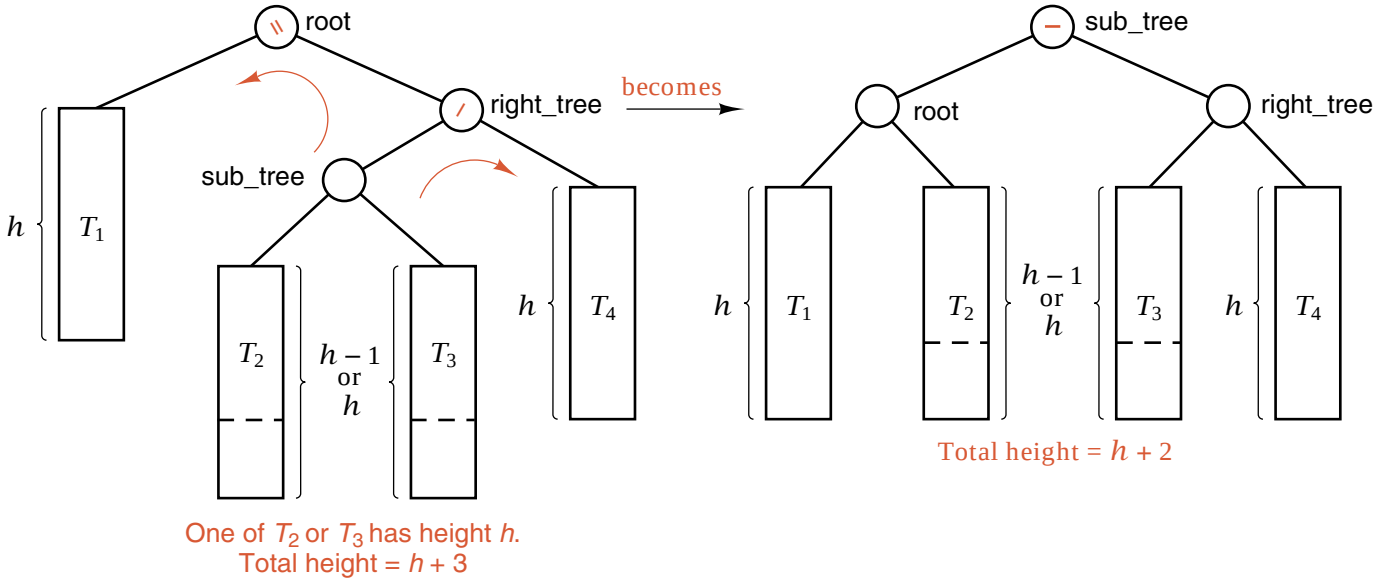
```
    else {                                  //    Insert in right subtree.
        result = avl_insert(sub_root->right, new_data, taller);
        if (taller ==  true)
            switch (sub_root->get_balance( )) {
            case left_higher:
                sub_root->set_balance(equal_height);
                taller = false;
                break;
            case equal_height:
                sub_root->set_balance(right_higher);
                break;
            case right_higher:
                right_balance(sub_root);
                taller = false;             //    Rebalancing always shortens the tree.
                break;
            }
    }
    return result;
}
```

Rotations of an AVL Tree




```
template <class Record>
void AVL_tree<Record> :: rotate_left(Binary_node<Record> * &sub_root)
/* Pre:  sub_root points to a subtree of the AVL_tree.  This subtree has a nonempty
         right subtree.
   Post: sub_root is reset to point to its former right child, and the former sub_root
         node is the left child of the new sub_root node.  */
{
  if (sub_root == NULL || sub_root->right == NULL)  //  impossible cases
    cout << "WARNING: program error detected in rotate_left" << endl;
  else {
    Binary_node<Record> *right_tree = sub_root->right;
    sub_root->right = right_tree->left;
    right_tree->left = sub_root;
    sub_root = right_tree;
  }
}
```

Double Rotation



The new balance factors for root and right_tree depend on the previous balance factor for sub_tree:

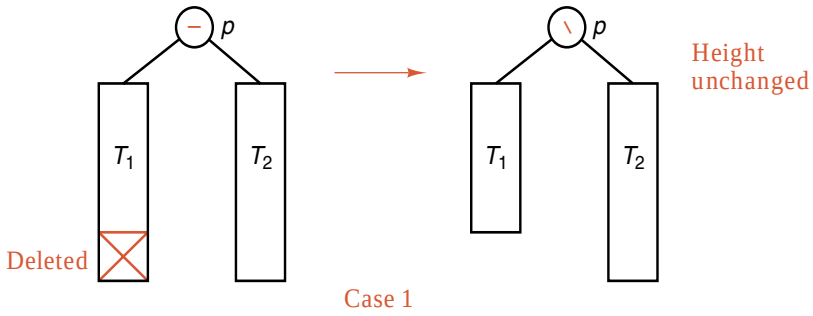
old sub_tree	*new* root	*new* right_tree	*new* sub_tree
–	–	–	–
/	–	\	–
\	/	–	–

```
template <class Record>
void AVL_tree<Record> ::
        right_balance(Binary_node<Record> * &sub_root)
/* Pre:   sub_root points to a subtree of an AVL_tree, doubly unbalanced on the right.
   Post:  The AVL properties have been restored to the subtree.
   Uses:  Methods of struct AVL_node; functions rotate_right, rotate_left.  */
{
   Binary_node<Record> * &right_tree = sub_root->right;
   switch (right_tree->get_balance( )) {
   case right_higher:                  //    single rotation left
      sub_root->set_balance(equal_height);
      right_tree->set_balance(equal_height);
      rotate_left(sub_root);  break;
   case equal_height:                  //    impossible case
      cout << "WARNING: program error in right_balance" << endl;
   case left_higher:                   //    double rotation left
      Binary_node<Record> *sub_tree = right_tree->left;
      switch (sub_tree->get_balance( )) {
      case equal_height:
         sub_root->set_balance(equal_height);
         right_tree->set_balance(equal_height);  break;
      case left_higher:
         sub_root->set_balance(equal_height);
         right_tree->set_balance(right_higher);  break;
      case right_higher:
         sub_root->set_balance(left_higher);
         right_tree->set_balance(equal_height);  break;
      }
      sub_tree->set_balance(equal_height);
      rotate_right(right_tree);
      rotate_left(sub_root);  break;
   }
}
```

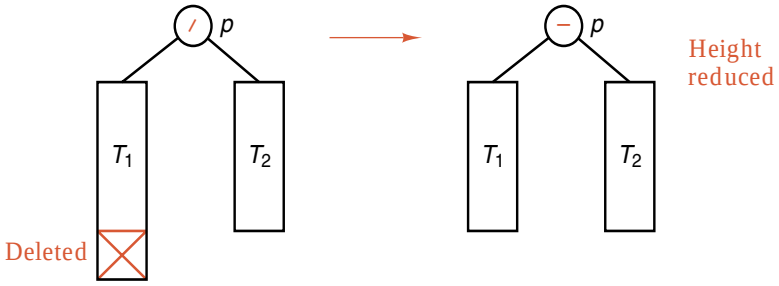
Removal of a Node

1. Reduce the problem to the case when the node x to be removed has at most one child.
2. Delete x. We use a **bool** variable shorter to show if the height of a subtree has been shortened.
3. While shorter is true do the following steps for each node p on the path from the parent of x to the root of the tree. When shorter becomes false, the algorithm terminates.
4. *Case 1*: Node p has balance factor equal. The balance factor of p is changed according as its left or right subtree has been shortened, and shorter becomes false.
5. *Case 2*: The balance factor of p is not equal, and the taller subtree was shortened. Change the balance factor of p to equal, and leave shorter as true.
6. *Case 3*: The balance factor of p is not equal, and the shorter subtree was shortened. Apply a rotation as follows to restore balance. Let q be the root of the taller subtree of p.
7. *Case 3a*: The balance factor of q is equal. A single rotation restores balance, and shorter becomes false.
8. *Case 3b*: The balance factor of q is the same as that of p. Apply a single rotation, set the balance factors of p and q to equal, and leave shorter as true.
9. *Case 3c*: The balance factors of p and q are opposite. Apply a double rotation (first around q, then around p), set the balance factor of the new root to equal and the other balance factors as appropriate, and leave shorter as true.

no rotations

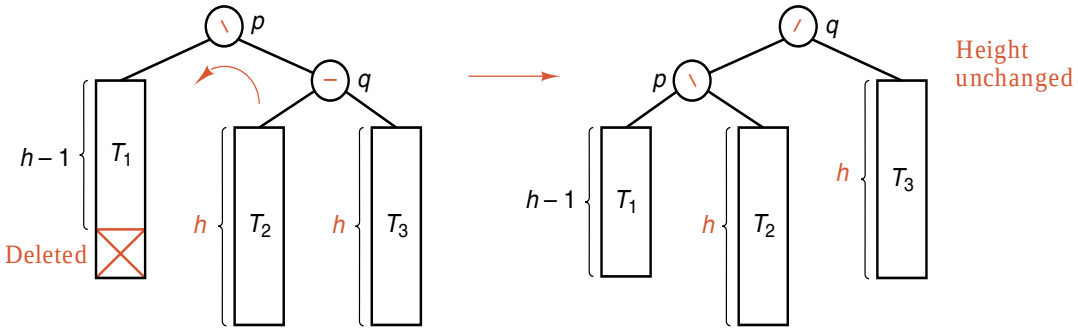


Case 1

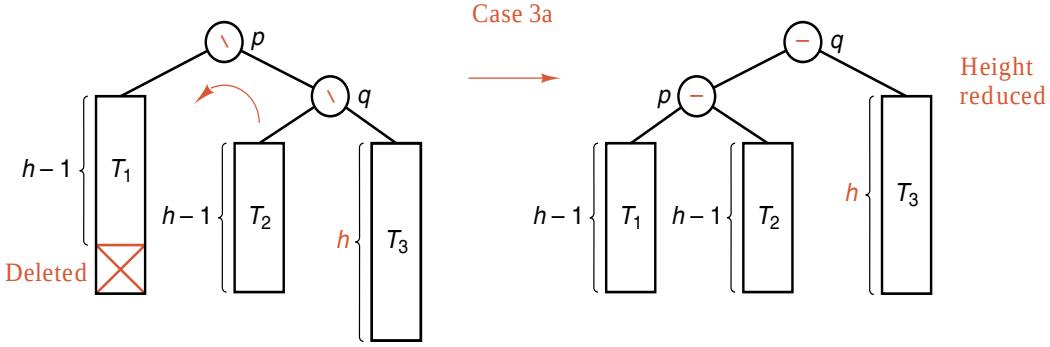


Case 2

single left rotations

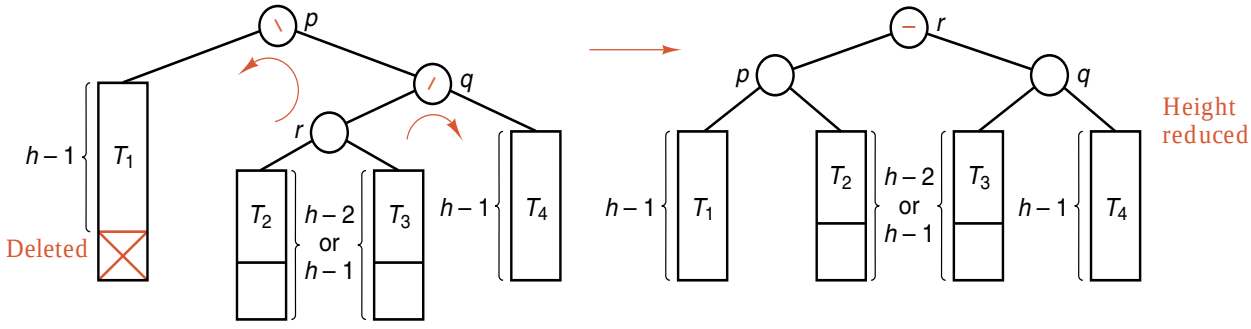


Case 3a

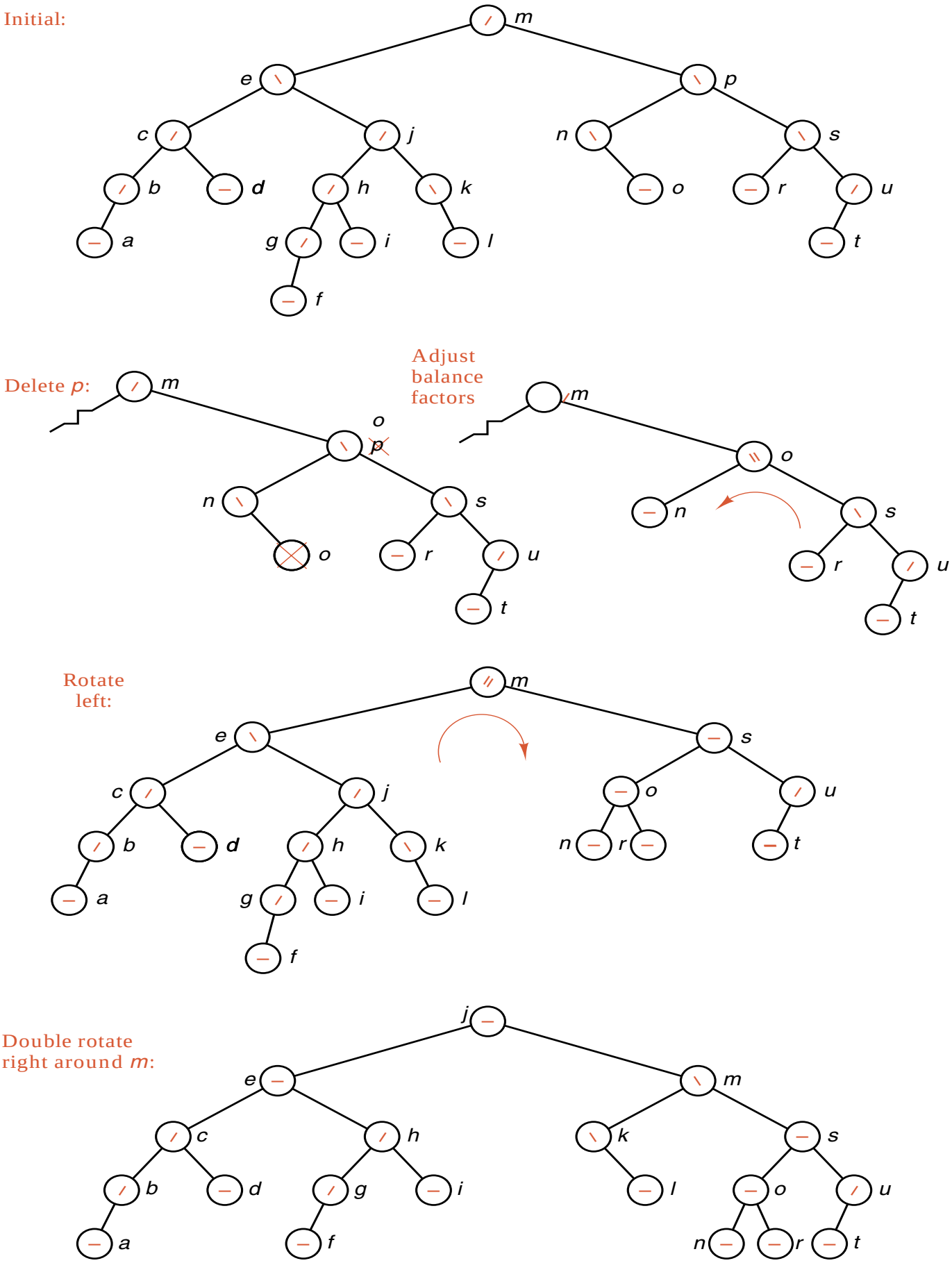


Case 3b

double rotation



Case 3c



Analysis of AVL Trees

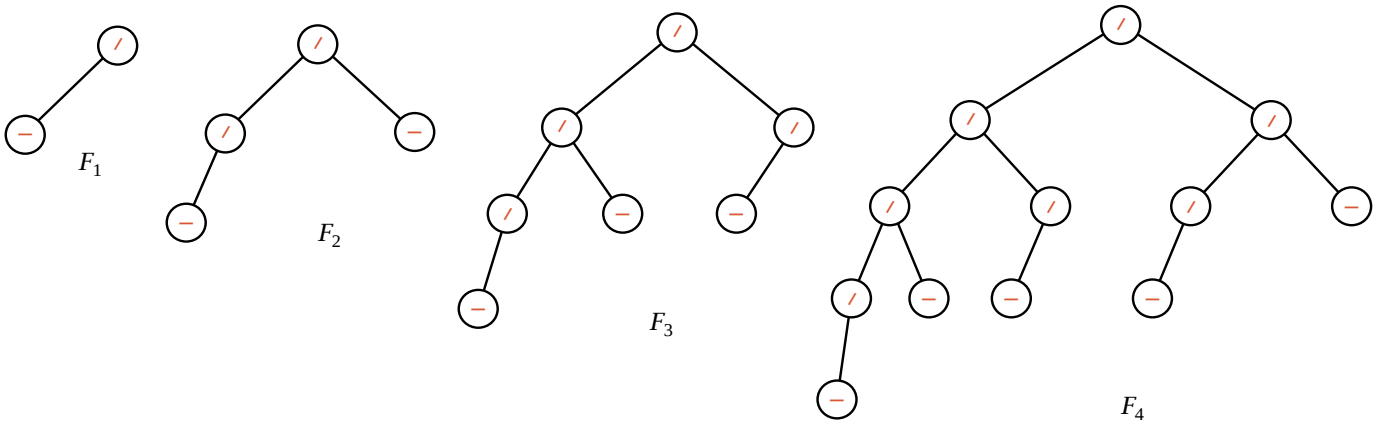
- The number of recursive calls to insert a new node can be as large as the height of the tree.
- At most one (single or double) rotation will be done per insertion.
- A rotation improves the balance of the tree, so later insertions are less likely to require rotations.
- It is very difficult to find the height of the *average* AVL tree, but the worst case is much easier. The worst-case behavior of AVL trees is essentially no worse than the behavior of random search trees.
- Empirical evidence suggests that the average behavior of AVL trees is much better than that of random trees, almost as good as that which could be obtained from a perfectly balanced tree.

Worst-Case AVL Trees

- To find the maximum height of an AVL tree with n nodes, we instead find the minimum number of nodes that an AVL tree of height h can have.

- Let F_h be such a tree, with left and right subtrees F_l and F_r. Then one of F_l and F_r, say F_l, has height $h - 1$ and the minimum number of nodes in such a tree, and F_r has height $h - 2$ with the minimum number of nodes.

- These trees, as sparse as possible for AVL trees, are called ***Fibonacci trees***.



Analysis of Fibonacci Trees

- If we write $|T|$ for the number of nodes in a tree T, we then have the recurrence relation for Fibonacci trees:

$$|F_h| = |F_{h-1}| + |F_{h-2}| + 1,$$

  where $|F_0| = 1$ and $|F_1| = 2$.

- By the evaluation of Fibonacci numbers in Section A.4,

$$|F_h| + 1 \approx \frac{1}{\sqrt{5}} \left[\frac{1 + \sqrt{5}}{2} \right]^{h+2}$$

- Take the logarithms of both sides, keeping only the largest terms:

$$h \approx 1.44 \lg |F_h|.$$

- The sparsest possible AVL tree with n nodes has height about $1.44 \lg n$ compared to:
  - A perfectly balanced binary search tree with n nodes has height about $\lg n$.
  - A random binary search tree, on average, has height about $1.39 \lg n$.
  - A degenerate binary search tree has height as large as n.

- Hence the algorithms for manipulating AVL trees are guaranteed to take no more than about 44 percent more time than the optimum. In practice, AVL trees do much better than this on average, perhaps as small as $\lg n + 0.25$.