Simple Regression

CS 700

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Acknowledgement

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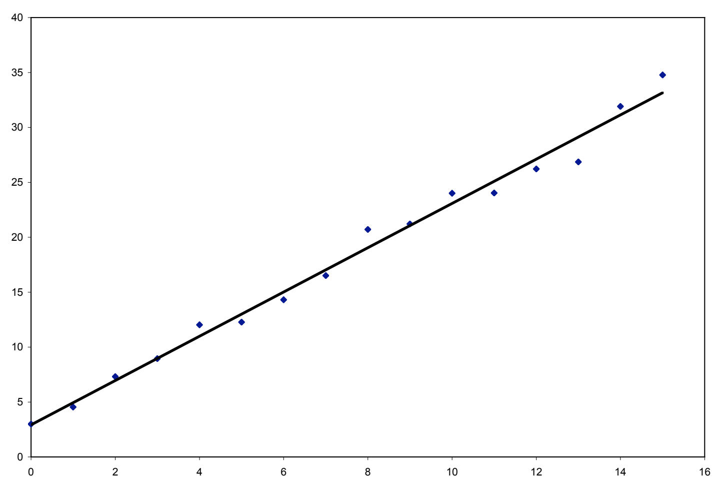
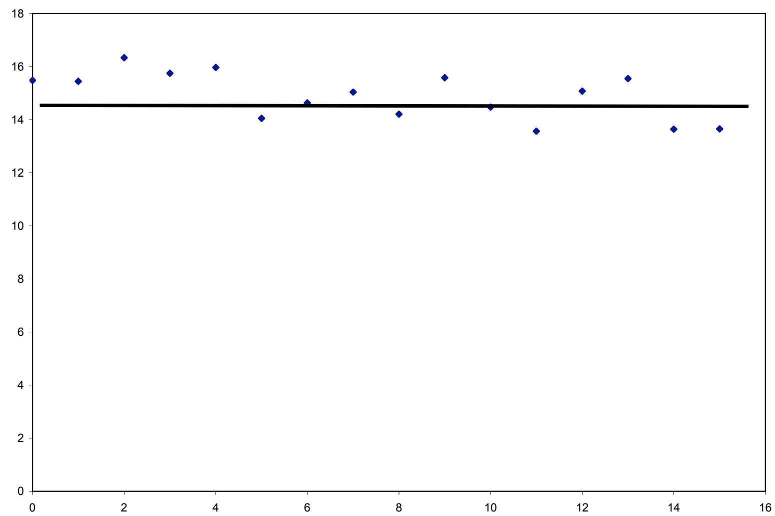
Basics

- Purpose of regression analysis: predict the value of a dependent or response variable from the values of at least one explanatory or independent variable (also called predictors or factors).
- Purpose of correlation analysis: measure the strength of the correlation between two variables.

Linear Relationship



No Relationship



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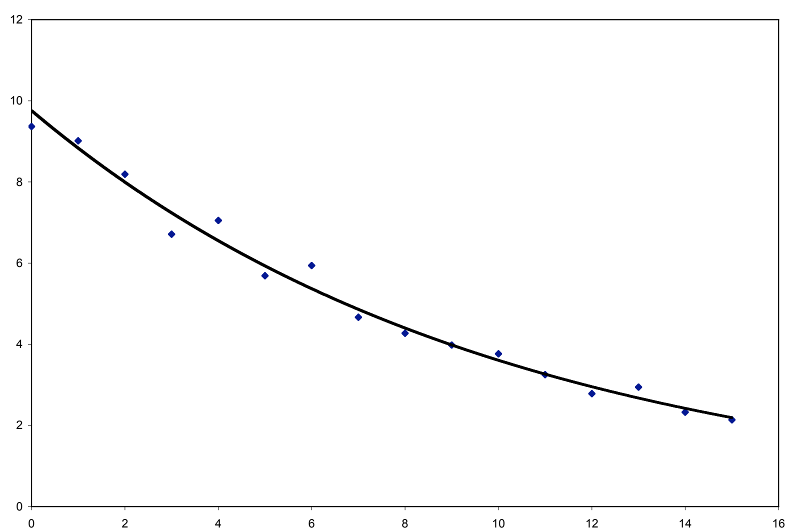
Negative Curvilinear



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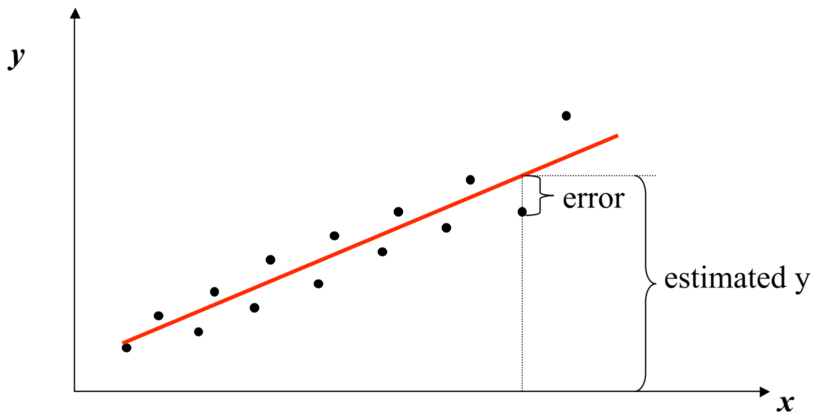
Simple Linear Regression
Residual Error



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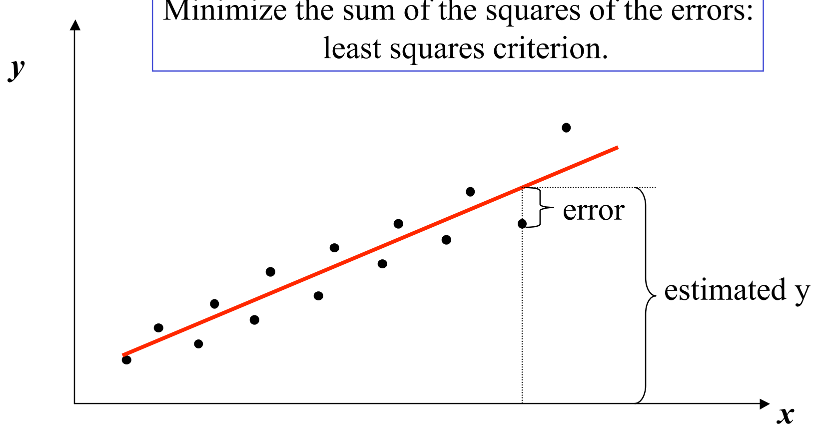
Simple Linear Regression
Selecting the "best" line

Minimize the sum of the squares of the errors: least squares criterion.



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Linear Regression

$$\hat{Y}_i = b_0 + b_1 X_i$$

$\hat{Y}_i$: predicted value of Y for observation i.

X_i : value of observation i.

b_0 and b_1 are chosen to minimize:

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} \left[Y_i - (b_0 + b_1 X_i)\right]^2$$

Subject to: $\sum_{i=1}^{n} e_i = 0$

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Method of Least Squares

$$b_1 = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{X}\overline{Y}}{\sum_{i=1}^{n} X_i^2 - n\left(\overline{X}\right)^2}$$

$$b_0 = \overline{Y} - b_1 \overline{X}$$

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Linear Regression Example

Number of I/Os (x)	CPU Time (y)	Estimate (0.0408*x +0.0508)	Error	Error Squared
1	0.092	0.092	0.0005	0.00000
2	0.134	0.132	0.0013	0.00000
3	0.165	0.173	-0.0083	0.00007
4	0.211	0.214	-0.0026	0.00001
5	0.242	0.255	-0.0128	0.00016
6	0.302	0.295	0.0067	0.00005
7	0.357	0.336	0.0206	0.00042
8	0.401	0.377	0.0239	0.00057
9	0.405	0.418	-0.0131	0.00017
10	0.442	0.459	-0.0161	0.00026
				0.00171

Xbar	5.5
Ybar	0.275
Sum x2	385
Sum xy	18.494616
b1	0.0408
b0	0.0508

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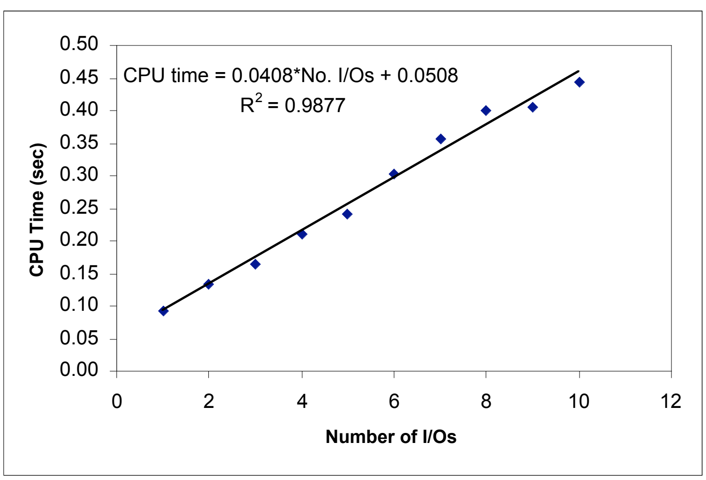
Linear Regression Example



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Allocation of Variation

- No regression model: use mean as predicted value. SSE is:

$$SST = \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

Sum of squares total

$$SSR = SST - SSE$$

Sum of squares explained by the regression.

Variation not explained by regression

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Allocation of Variation

- Coefficient of determination (R^2): fraction of variation explained by the regression.

$$R^2 = \frac{SSR}{SST} = \frac{SST - SSE}{SST} = 1 - \frac{SSE}{SST}$$

The closer R^2 is to one, the better is the regression model.

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Number of I/Os (x)	CPU Time (y)	Estimate (0.0408*x +0.0508)	Error	Error Squared	SSY
1	0.092	0.092	0.0005	0.00000	0.00848
2	0.134	0.132	0.0013	0.00000	0.017882
3	0.165	0.173	-0.0084	0.00007	0.027173
4	0.211	0.214	-0.0027	0.00001	0.044645
5	0.242	0.255	-0.0129	0.00017	0.058505
6	0.302	0.296	0.0066	0.00004	0.091331
7	0.357	0.336	0.0204	0.00042	0.127331
8	0.401	0.377	0.0238	0.00056	0.160771
9	0.405	0.418	-0.0133	0.00018	0.163795
10	0.442	0.459	-0.0163	0.00027	0.195783
	0.275			0.00172	0.89570

SST	0.1388841
SSR	0.1371690
R2	0.9876514

SSE SSY

$$SST = \sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \left(\sum_{i=1}^{n} Y_i^2\right) - n\bar{Y}^2 = SSY - SS0$$

$$SSE = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

$$SSR = \sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2 = SST - SSE$$

$$R^2 = \frac{SSR}{SST}$$ coefficient of determination.

The higher the value of R^2 the better the regression.

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Standard Deviation of Errors

- Variance of errors: divide the sum of squares (SSE) by the number of degrees of freedom (n-2 since two regression parameters need to be computed first).

$$s_e^2 = \frac{SSE}{n-2}$$ ← Mean squared error (MSE)

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Degrees of freedom of various sum of squares.

SST	n-1	Need to compute $\overline{Y}$
SSY	n	Does not depend on any other parameter
SS0	1	
SSE	n-2	Need to compute two regression parameters
SSR	1	=SST-SSE

Degrees of freedom add as sum of squares do.

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Confidence Interval for Regression Parameters

- b_0 and b_1 were computed from a sample. So, they are just estimates of the true parameters β_0 and β_1 for the true model.
- Standard deviations for b_0 and b_1.

$$s_{b_0} = s_e \sqrt{\frac{1}{n} + \frac{(\overline{X})^2}{\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2}}$$

$$s_{b_1} = \frac{s_e}{\sqrt{\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2}}$$

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Confidence Interval for Regression Parameters

100(1-α)% confidence interval for b_o and b_1

$$b_0 \pm t_{[1-\alpha/2;n-2]} s_{b_0}$$

$$b_1 \pm t_{[1-\alpha/2;n-2]} s_{b_1}$$

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Confidence Interval Example

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2	0.134	0.132	0.0013	0.00000
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7	0.357	0.336	0.0206	0.00042
8	0.401	0.377	0.0239	0.00057
9	0.405	0.418	-0.0131	0.00017
10	0.442	0.459	-0.0161	0.00026
			SSE:	0.00171

Xbar	5.5		
Ybar	0.275		
Sum x2	385		
Sum xy	18.494616		
b1	0.0408		
b0	0.0508		
se^2	0.0002144	Lower bo	0.027772
se	0.0146411	Upper bo	0.073900
sb0	0.0100017		
sb1	0.0016119	Lower b1	0.037058576
95% confidence level		Upper b1	0.044492804
alpha	0.05		
t[1-alpha/2;n-2]	2.3060056		
SST	0.1388841		
SSR	0.13717		
R2	0.9876524		

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Confidence Interval for the Predicted Value

- The standard deviation of the mean of a future sample of *m* observations at $X = X_p$ is

$$s_{\hat{y}_{mp}} = s_e \left[\frac{1}{m} + \frac{1}{n} + \frac{\left(X_p - \overline{X}\right)^2}{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2} \right]^{1/2}$$

As the future sample size (*m*) increases, the standard deviation for predicted value decreases.

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Confidence Interval for the Predicted Value

100(1-α)% confidence interval for the predicted value for a future sample of size *m* at X_p:

$$\hat{y}_p \pm t_{[1-\alpha/2;n-2]} s_{\hat{y}_{mp}}$$

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Linear Regression Assumptions

- Linear relationship between the response (y) and the predictor (x).
- The predictor (x) is non-stochastic and is measured without any error.
- Errors are statistically independent.
- Errors are normally distributed with zero mean and a constant standard deviation.

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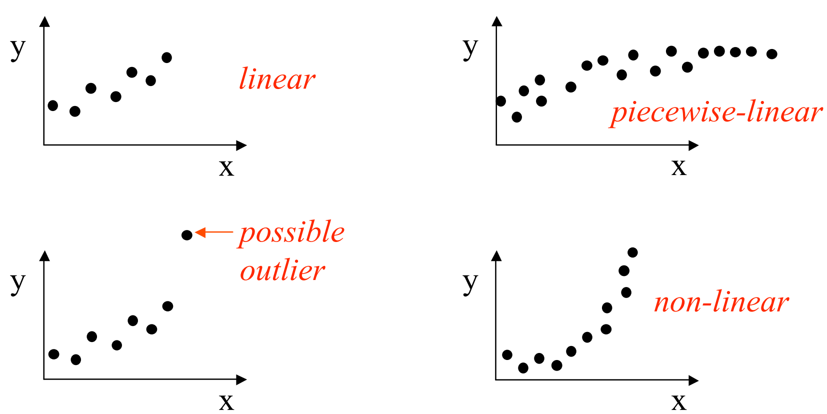
Linear Regression Assumptions

Linear relationship between the response (y) and the predictor (x).



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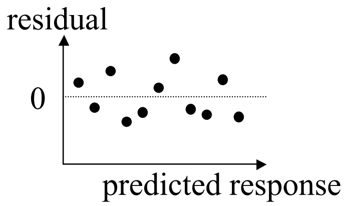
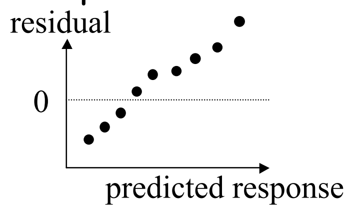
Linear Regression Assumptions

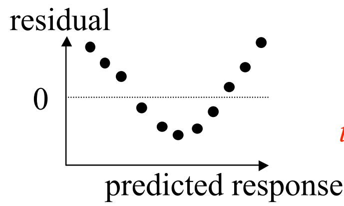
Errors are statistically independent.



no trend



trend



trend

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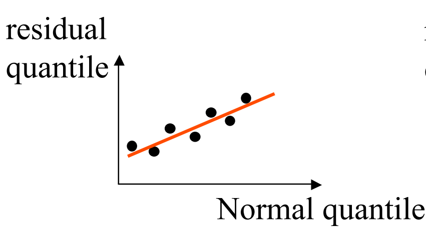
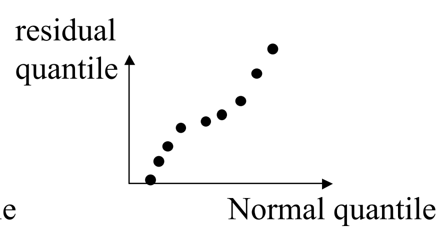
Linear Regression Assumptions

Errors are normally distributed.



normally distributed errors



non-normally distributed errors

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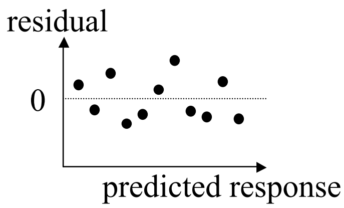
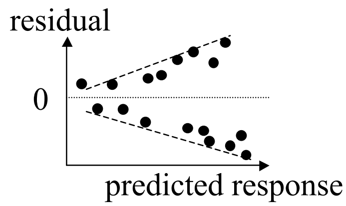
Linear Regression Assumptions

Errors have a constant standard deviation.



no trend in spread



increasing spread

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Other Regression Models

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Multiple Linear Regression

- Use to predict the value of the response variable as function of k predictor variables $x_1, \ldots, x_n$.

$$\hat{Y}_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + \ldots + b_x X_{ki}$$

- Similar to simple linear regression.
- MS Excel can be used to do multiple linear regression.

CPU Time (yi)	I/O Time (x1i)	Memory Requirement (x2i)
2	14	70
5	16	75
7	27	144
9	42	190
10	39	210
13	50	235
20	83	400

Want to find:

$$\text{CPUTime} = b_0 + b_1 * \text{I/OTime} + b_2 * \text{MemoryRequirement}$$

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.9870 ← R
R Square	0.9742
Adjusted R Square	0.9614
Standard Error	1.1511
Observations	7

	Coefficients	*Standard Error*	*t Stat*	*Lower 95%*	*Upper 95%*	*Lower 90.0%*	*Upper 90.0%*
Intercept (b0)	-0.16145	0.91345	-0.17674	-2.69759	2.37470	-2.10878	1.78589
X Variable 1 (b1)	0.11824	0.19260	0.61389	-0.41652	0.65299	-0.29236	0.52884
X Variable 2 (b2)	0.02650	0.04045	0.65519	-0.08580	0.13881	-0.05973	0.11273

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Curvilinear Regression

Approach: plot a scatter plot. If it does not look linear, try non-linear models:

Non-linear	Linear
$y = a + b / x$	$y = a + b(1/x)$
$y = 1/(a + bx)$	$(1/y) = a + bx$
$y = x/(a + bx)$	$(x/y) = a + bx$
$y = a \times b^x$	$\ln y = \ln a + x \ln b$
$y = a + bx^n$	$y = a + b(x^n)$

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