2^{k-p} Fractional Factorial Designs

Fractional Factorial Designs

- If we have 7 factors, a 2^7 factorial design will require 128 experiments
- How much information can we obtain from fewer experiments, e.g. $2^{7-4} = 8$ experiments?
- A 2^{k-p} design allows the analysis of k two-level factors with fewer experiments

A 2^{7-4} Experimental Design

Consider the 2^3 design below:

Experiment #	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
2	1	1	-1	-1	-1	-1	1	1
3	1	-1	1	-1	-1	1	-1	1
4	1	1	1	-1	1	-1	-1	-1
5	1	-1	-1	1	1	-1	-1	1
6	1	1	-1	1	-1	1	-1	-1
7	1	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1	1

If the factors, AB, AC, BC, ABC are replaced by D, E, F, and G we get a 2^{7-4} design

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A 2^{7-4} design

If the interactions AB, AC, AD,…, ABCD are negligible we can use the table below

Experiment #	I	A	B	C	D	E	F	G	y
1	1	-1	-1	-1	1	1	1	-1	20
2	1	1	-1	-1	-1	-1	1	1	35
3	1	-1	1	-1	-1	1	-1	1	7
4	1	1	1	-1	1	-1	-1	-1	42
5	1	-1	-1	1	1	-1	-1	1	36
6	1	1	-1	1	-1	1	-1	-1	50
7	1	-1	1	1	-1	-1	1	-1	45
8	1	1	1	1	1	1	1	1	82
Total	317	101	35	109	43	1	47	3	
Total/8	39.62	12.62	4.37	13.62	5.37	0.12	5.9	0.37	
Percent variation		37.26	4.74	43.4	6.75	0	8.1	0.03	

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Preparing the sign table for a 2^{k-p} design

1. Choose k-p factors and prepare a complete sign table for a full factorial design with k-p factors.
   - There are 2^{k-p} rows and columns in the table.
   - The first column is marked I and consists of all 1's.
   - The next k-p columns correspond to the k-p selected factors. The remaining columns correspond to the products of these factors.
2. Of the 2^{k-p}-k+p-1 remaining columns, select p columns corresponding to the p factors that were not chosen in step 1.

   Note: there are several possibilities; the columns corresponding to negligible interactions should be chosen.

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A 2^{4-1} design

Experiment #	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
2	1	1	-1	-1	-1	-1	1	1
3	1	-1	1	-1	-1	1	-1	1
4	1	1	1	-1	1	-1	-1	-1
5	1	-1	-1	1	1	-1	-1	1
6	1	1	-1	1	-1	1	-1	-1
7	1	-1	1	1	-1	-1	1	-1
8	1	1	1	1	1	1	1	1

If the ABC interaction is negligible, we should replace ABC with D. If AB is negligible, we can replace AB with D.

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Confounding

- The drawback of 2^{k-p} designs is that the experiments only yield the combined effects of two or more factors. This is called confounding
  - On the previous slide, the effects of ABC and D are confounded (denoted as ABC = D)
- In a 2^{k-1} design, every column represents a sum of two effects.
  - For our example,
    - A = BCD, B = ACD, C = ABD, AB = CD, AC = BD, BC = AD, ABC = D, I = ABCD
    - This means that columns for A, B, C, D actually correspond to A + BCD, B + ACD, C + ABD, D+ ABC, etc.
  - If we replace AB with D,
    - I = ABD, A = BD, B = AD, C = A BCD, D = AB, AC = BCD, BC = ACD, ABC = CD
- In a 2^{k-p} design, 2^p effects are confounded

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Algebra of Confounding

Consider the first design in which ABC is replaced with D

- Here, I = ABCD
- All the confoundings can be generated using the following rules
  1. I is treated as unity.e.g. I multiplied by A is A
  2. Any term with a power of 2 is erased, e.g. AB^2C is the same as AC.

The polynomial I = ABCD is used to generate all the confoundings for this design, and is called the generator polynomial

The second design in which AB was replaced by D in the sign table has generator polynomial I = ABD

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Design Resolution

- The resolution of a design is measured by the order of effects that are confounded
  - The effect ABCD is of order 4, while I is of order 0
  - If an *i*-th order effect is confounded with a *j*-th order term, the confounding is of order *i+j*
  - The minimum of orders of all confoundings of a design is called its resolution
    - We can easily determine the resolution of a design by looking at the generator polynomial, e.g. if I = ABCD, then the design has resolution 4, if I = ABD, the design has resolution 3
- In general, higher resolution designs are considered better under the assumption that higher order interactions are smaller than lower-order effects

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One Factor Experiment Design

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One Factor Experiment Design

- So far: unlimited factors, two levels
- Now: unlimited levels, one factor

Model:

$y_{ij} = \mu + \alpha_j + e_{ij}$, where y_{ij} is the ith response with the factor at level j, μ is the mean response, α_j is the effect of alternative j, and e_{ij} is the error term

The effects are computed such that

$$\sum \alpha_j = 0, \qquad \sum_i e_{ij} = 0, \qquad \sum_j \sum_i e_{ij} = 0$$

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Model cont'd

Notation: j (factor level), i (replication), a (number of levels), r (number of replicas)

Example: $a = 3, r = 5$

R	V	Z
144	101	130
120	144	180
176	211	141
288	288	374
144	72	302

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Model cont'd

Notation:

$\bar{y}_{..} = \mu$, grand mean (avg. of all responses i,j)

$\bar{y}_{.j} = \mu + \alpha_j$, column mean (avg of all responses for a particular factor level)

	R	V	Z	
	144	101	130	
	120	144	180	
	176	211	141	
	288	288	374	
	144	72	302	
Column Sum	872	816	1127	2815
Column mean	174.4 $\bar{y}_{.1}$	163.2	225.4	187.7 $\bar{y}_{..}$
Column effect	-13.3 α_1	-24.5	37.7	

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Estimating experimental errors

Estimated response for the *j*th alternative is

$$\hat{y}_j = \mu + \alpha_j$$

$$e_{ij} = y_{ij} - \hat{y}_j$$

$$SSE = \sum_{i=1}^{r}\sum_{j=1}^{a} e_{ij}^2$$

For the example on previous slide,

SSE = $(144 - 187.7 + 13.3)^2 + (101 - 187.7 + 24.5)^2 + (130 - 187.7 - 37.7)^2 + \ldots + (302 - 187.7 - 37.7)^2$

= 94,365.20

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Allocation of variation

- We can show that SSY = SS0 + SSA + SSE, where SSY is sum of squares of y, SS0 is the sum of squares of the grand mean, SSA is the sum of squares of effects, and SSE is the sum of squares of errors
- SSE can be easily calculated from SSY, SS0, and SSA

$$SS0 = ar\mu^2,\ SSA = r\sum_{j=1}^{a}\alpha_j^2$$

- Further, SST = SSY - SS0 = SSA + SSE
- In our example, SST = 105,357; SSA = 10,992 (10.4%) ; SSE = 94,365 (89.6%)
  - Is SSA statistically significant?

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Analysis of variance

- Allocation of variation shows that almost 90% of the variation is due to SSE
- In general, relatively high SSE could mean
  - Factor under consideration is not important
  - Number of replicas is much larger than number of factor levels
  - Maybe we have two few samples and a "bad" sample with high errors
- Statistical procedure for analysis of significance of various factors -- Analysis of Variance (ANOVA)

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ANOVA

- Consider that

  $SSY = SS0 + SSA + SSE$

  $ar = 1 + (a-1) + a(r-1)$ Degrees of freedom

- Define

  $MSA = SSA/(a-1)$ Mean Square of A

  $MSE = SSE/a(r-1)$ Mean Square of E

- Then, the ratio MSA/MSE has a F-distribution with $(a-1)$ numerator degrees of freedom and $a(r-1)$ denominator degrees of freedom
  - *F(n,m) denotes F distribution where n and m and numerator and denominator degrees of freedom respectively*

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F-test

- Tests following null hypothesis:

  Response variable does not depend upon any factor α

  Acceptance criteria: MSA/MSE ratio does not exceed the 1-α quantile of F distribution

- So the question: is the factor statistically significant is equivalent to rejecting the null hypotheis above
  - In other words, the F-statistic from our data should exceed the theoretical F value

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F-test example

For our example,

MSA = SSA/(a-1) = 10,992/2 = 5496.1

MSE = SSE/a(r-1) = 94,265/12 = 7863.8

Computed F statistic = 5496.1/7963.8 = 0.7

Theoretical F(0.9;2,12) = 2.8

Thus, we can conclude that the factor under consideration is not statistically significant

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Additional Reading

- Visual Diagnostic Tests for verifying assumptions - Section 20.6
- Confidence intervals for Effects - Section 20.7
- Unequal sample sizes - Section 20.8

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Two factor full factorial designs without replications

- Experiment design with two factors, each of which can have an arbitrary number of levels
  - Initially, we will not consider replications
  - Factors A and B, with number of levels *a* and *b* respectively, number of experiments is *ab*

Model

$y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}$, where y_{ij} is the observed response with the factor A at level *j* and the factor B at level *i*,
μ is the mean response, α_j is the effect of factor A at level *j*, and β_i is the effect of factor B at level *i*, and e_{ij} is the error term

The effects are computed such that

$$\sum \alpha_j = 0, \quad \sum \beta_i = 0, \quad \sum_j \sum_i e_{ij} = 0$$

We obtain $\bar{y}_{..} = \mu, \quad \bar{y}_{i.} = \mu + \beta_i, \quad \bar{y}_{.j} = \mu + \alpha_j$

Computation of Effects

Workload	Two Caches	One Cache	No Cache	Row Sum	Row Mean	Row Effect
ASM	54.0	55.0	106.0	215.0	71.7	-0.5 β_1
TECO	60.0	60.0	123.0	243.0	81.0	8.8
SIEVE	43.0	43.0	120.0	206.0	68.7	-3.5
DHRYSTONE	49.0	52.0	111.0	212.0	70.7	-1.5
SORT	49.0	50.0	108.0	207.0	69.0	-3.2
Column Sum	255.0	260.0	568.0	1083.0		
Column Mean	51.0	52.0	113.6		72.2	$\bar{y}_{..}$
Column effect	-21.2 α_1	-20.2	41.4			

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Estimating experimental errors

$$\hat{y}_j = \mu + \alpha_j + \beta_i$$

$$e_{ij} = y_{ij} - \hat{y}_j = y_{ij} - \mu - \alpha_j - \beta_i$$

$$SSE = \sum_{i=1}^{b}\sum_{j=1}^{a} e_{ij}^2$$

For our example, $\hat{y}_{11} = 72.2 - 21.2 - 0.5 = 50.5$

$e_{11} = 54 - 50.5 = 3.5$

$$SSE = 3.5^2 + 0.2^2 + ... + (-2.4)^2 = 236.80$$

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Allocation of variation

- We can show SSY = SS0 + SSA + SSB + SSE and SST = SSY - SS0 = SSA+SSB + SSE
- For our example, SSY = 91,595; SS0 = 78,192.6; SSA = 12,857.2, SSB = 308.4, SST = 13,402.41 SSE = SST - SSA - SSB = 236.8
- The percentage of variation explained by the cache is 95.9%, due to workloads is 2.3% and the unexplained variation is 1.8%

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Analysis of variance (ANOVA)

- Similar to one-factor analysis

  MSA = SSA/(a-1); MSB = SSB/(b-1)

  MSE = SSE/(a-1)(b-1)
- F-ratio for factor A is F_A = MSA/MSE and for factor B is F_B = MSB/MSE
  - If greater than theoretical F-value then factor is statistically significant
  - For our example, F_A = 217.2, F_B = 2.6, theoretical F value = 2.8, so the first factor (cache) is statistically significant

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Additional Reading

- Section 21.6 - confidence intervals for effects
- Section 21.7 - multiplicative models for two factor experiments
- Section 21.8 - handling missing observations (optional)

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Two-factor full factorial design with replications

$y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ij}$, where y_{ijk} is the observed response in the kth replication of the experiment with the factor A at level j and the factor B at level i, μ is the mean response, α_j is the effect of factor A at level j, β_i is the effect of factor B at level i, γ_{ij} is the effect of interaction between factor A at level j and factor B at level i, and and e_{ij} is the experimental error

The effects are computed such that

$$\sum \alpha_j = 0, \qquad \sum \beta_{\mathrm{i}} = 0,$$

The interactions are computed so that their row as well as column sums are 0

The errors in each experiment add to 0

$$\sum_{k=1}^{r} e_{ijk} = 0, \forall i, j$$

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Computation of effects

- The observations are arranged in b rows and a columns with each cell containing r observations
- Compute the average of the r observations for each cell
- Then, proceed as in the analysis of two-factor design without replication

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Computation of errors

$$\hat{y}_{ij} = \mu + \alpha_j + \beta_i + \gamma_{ij} = \bar{y}_{ij.}$$ (the average of the r observations in a cell)

The error in the kth replication of the experiment is $e_{ijk} = y_{ijk} - \bar{y}_{ij.}$

$$SSE = \sum_{i=1}^{b}\sum_{j=1}^{a}\sum_{k=1}^{r} e_{ijk}^2$$

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Allocation of variation and ANOVA

- We can show

  SST = SSY - SS0 = SSA + SSB + SSAB + SSE

- ANOVA
  - F-test: compute MSA/MSE, MSB/MSE, MSAB/MSE
  - Degrees of freedom: SSA has a-1, SSB has b-1, SSAB has (a-1)(b-1), SSE has ab(r-1)

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Further Reading

- Section 22.6 - confidence intervals for effects
- Chapter 23 - General Full factorial designs with k factors
  - Generalization of analysis techniques discussed
  - Informal (non-statistical methods) for determining the important factors

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