2^{k-p} Fractional Factorial Designs

- If we have 7 factors, a $2^7$ factorial design will require 128 experiments.
- How much information can we obtain from fewer experiments, e.g. $2^{7-4} = 8$ experiments?
- A $2^{k-p}$ design allows the analysis of $k$ two-level factors with fewer experiments.
A 2^7-4 Experimental Design

Consider the 2^3 design below:

<table>
<thead>
<tr>
<th>Experiment #</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

If the factors, AB, AC, BC, ABC are replaced by D, E, F, and G we get a 2^7-4 design

A 2^7-4 design

If the interactions AB, AC, AD, ..., ABCD are negligible we can use the table below:

<table>
<thead>
<tr>
<th>Experiment #</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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</tbody>
</table>

| Total         | 317 | 101 | 35 | 109 | 43 | 1  | 47 | 3  |
| Total/8       | 39.62 | 12.62 | 4.37 | 13.62 | 5.37 | 0.12 | 5.9 | 0.37 |
| Percent variation | **37.26** | 4.74 | **43.4** | 6.75 | 0  | 8.1 | 0.03 |
Preparing the sign table for a $2^{k-p}$ design

1. Choose $k-p$ factors and prepare a complete sign table for a full factorial design with $k-p$ factors.
   There are $2^{k-p}$ rows and columns in the table.
   The first column is marked I and consists of all 1's.
   The next $k-p$ columns correspond to the $k-p$ selected factors. The remaining columns correspond to the products of these factors.

2. Of the $2^{k-p}-k+p-1$ remaining columns, select $p$ columns corresponding to the $p$ factors that were not chosen in step 1.
   Note: there are several possibilities; the columns corresponding to negligible interactions should be chosen.

A $2^{4-1}$ design

<table>
<thead>
<tr>
<th>Experiment #</th>
<th>I</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
<th>ABC</th>
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<td>1</td>
</tr>
</tbody>
</table>

If the ABC interaction is negligible, we should replace ABC with D. If AB is negligible, we can replace AB with D.
Confounding

- The drawback of $2^k$ designs is that the experiments only yield the combined effects of two or more factors. This is called confounding
  - On the previous slide, the effects of $ABC$ and $D$ are confounded (denoted as $ABC = D$)
- In a $2^{k-1}$ design, every column represents a sum of two effects.
  - For our example,
    - $A = BCD$, $B = ACD$, $C = ABD$, $AB = CD$, $AC = BD$, $BC = AD$, $ABC = D$, $I = ABCD$
    - This means that columns for $A$, $B$, $C$, $D$ actually correspond to $A + BCD$, $B + ACD$, $C + ABD$, $D + ABC$, etc.
  - If we replace $AB$ with $D$,
    - $I = ABD$, $A = BD$, $B = AD$, $C = A BCD$, $D = AB$, $AC = BCD$, $BC = ACD$, $ABC = CD$
- In a $2^{k-p}$ design, $2^p$ effects are confounded

Algebra of Confounding

Consider the first design in which $ABC$ is replaced with $D$
- Here, $I = ABCD$
- All the confoundings can be generated using the following rules
  1. $I$ is treated as unity, e.g., $I$ multiplied by $A$ is $A$
  2. Any term with a power of 2 is erased, e.g., $AB^3C$ is the same as $AC$.

The polynomial $I = ABCD$ is used to generate all the confoundings for this design, and is called the generator polynomial.

The second design in which $AB$ was replaced by $D$ in the sign table has generator polynomial $I = ABD$
Design Resolution

- The resolution of a design is measured by the order of effects that are confounded
  - The effect ABCD is of order 4, while I is of order 0
  - If an \(i\)-th order effect is confounded with a \(j\)-th order term, the confounding is of order \(i+j\)
  - The minimum of orders of all confoundings of a design is called its resolution
    - We can easily determine the resolution of a design by looking at the generator polynomial, e.g. if I = ABCD, then the design has resolution 4, if I = ABD, the design has resolution 3

- In general, higher resolution designs are considered better under the assumption that higher order interactions are smaller than lower-order effects

One Factor Experiment Design
One Factor Experiment Design

- So far: unlimited factors, two levels
- Now: unlimited levels, one factor

Model:

\[ y_{ij} = \mu + \alpha_j + e_{ij}, \]  
where \( y_{ij} \) is the \( i \)th response with the factor at level \( j \), \( \mu \) is the mean response, \( \alpha_j \) is the effect of alternative \( j \), and \( e_{ij} \) is the error term.

The effects are computed such that

\[ \sum \alpha_j = 0, \quad \sum_i e_{ij} = 0, \quad \sum_j \sum_i e_{ij} = 0. \]

Model cont’d

Notation: \( j \) (factor level), \( i \) (replication), \( a \) (number of levels), \( r \) (number of replicas).

Example: \( a = 3, r = 5 \)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>V</td>
<td>Z</td>
</tr>
<tr>
<td>---</td>
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<td>---</td>
</tr>
<tr>
<td>144</td>
<td>101</td>
<td>130</td>
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<tr>
<td>120</td>
<td>144</td>
<td>180</td>
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<tr>
<td>176</td>
<td>211</td>
<td>141</td>
</tr>
<tr>
<td>288</td>
<td>288</td>
<td>374</td>
</tr>
<tr>
<td>144</td>
<td>72</td>
<td>302</td>
</tr>
</tbody>
</table>
**Model cont’d**

Notation:
- $\bar{y}_i = \mu$, grand mean (avg. of all responses $i,j$)
- $\bar{y}_{ij} = \mu + \alpha_j$, column mean (avg of all responses for a particular factor level)

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>V</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>144</td>
<td>101</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>144</td>
<td>180</td>
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</tr>
<tr>
<td>176</td>
<td>211</td>
<td>141</td>
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<td>288</td>
<td>288</td>
<td>374</td>
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</tr>
<tr>
<td>144</td>
<td>72</td>
<td>302</td>
<td></td>
</tr>
<tr>
<td>Column Sum</td>
<td>872</td>
<td>816</td>
<td>1127</td>
</tr>
<tr>
<td>Column mean</td>
<td>174.4</td>
<td>$\bar{y}_{.1}$</td>
<td>163.2</td>
</tr>
<tr>
<td>Column effect</td>
<td>-13.3</td>
<td>$\alpha_{.1}$</td>
<td>-24.5</td>
</tr>
</tbody>
</table>

**Estimating experimental errors**

Estimated response for the $j$th alternative is

$$\hat{y}_j = \mu + \alpha_j$$

$$e_{ij} = y_{ij} - \hat{y}_j$$

$$SSE = \sum_{i=1}^{r} \sum_{j=1}^{a} e_{ij}^2$$

For the example on previous slide,

$$SSE = (144 - 187.7 + 13.3)^2 + (101 - 187.7 + 24.5)^2 + (130 - 187.7 - 37.7)^2 + \ldots + (302 - 187.7 - 37.7)^2$$

$$= 94,365.20$$
**Allocation of variation**

- We can show that $SSY = SS0 + SSA + SSE$, where $SSY$ is the sum of squares of $y$, $SS0$ is the sum of squares of the grand mean, $SSA$ is the sum of squares of effects, and $SSE$ is the sum of squares of errors.
- $SSE$ can be easily calculated from $SSY$, $SS0$, and $SSA$:
  \[
  SS0 = ar\mu^2, \quad SSA = r \sum_{j=1}^{a} \alpha_j^2
  \]
- Further, $SST = SSY - SS0 = SSA + SSE$.
- In our example, $SST = 105,357$; $SSA = 10,992 (10.4\%)$; $SSE = 94,365 (89.6\%)$.
  - Is $SSA$ statistically significant?

---

**Analysis of variance**

- Allocation of variation shows that almost 90% of the variation is due to $SSE$.
- In general, relatively high $SSE$ could mean:
  - Factor under consideration is not important.
  - Number of replicas is much larger than number of factor levels.
  - Maybe we have too few samples and a "bad" sample with high errors.
- Statistical procedure for analysis of significance of various factors -- Analysis of Variance (ANOVA).
**ANOVA**

- Consider that
  \[ SSY = SS0 + SSA + SSE \]
  \[ ar = 1 + (a - 1) + a (r-1) \quad \text{Degrees of freedom} \]

- Define
  \[ MSA = SSA/(a-1) \quad \text{Mean Square of A} \]
  \[ MSE = SSE/a(r-1) \quad \text{Mean Square of E} \]

- Then, the ratio \( MSA/MSE \) has a F-distribution with \( (a-1) \) numerator degrees of freedom and \( a(r-1) \) denominator degrees of freedom
  \[ F(n,m) \text{ denotes F distribution where n and m and numerator and denominator degrees of freedom respectively} \]

---

**F-test**

- Tests following null hypothesis:
  Response variable does not depend upon any factor \( \alpha \)
  Acceptance criteria: \( MSA/MSE \) ratio does not exceed the \( 1-\alpha \) quantile of F distribution

- So the question: is the factor statistically significant is equivalent to rejecting the null hypothesis above
  \[ \text{In other words, the F-statistic from our data should exceed the theoretical F value} \]
**F-test example**

For our example,

\[ MSA = \frac{SSA}{(a-1)} = \frac{10,992}{2} = 5496.1 \]
\[ MSE = \frac{SSE}{a(r-1)} = \frac{94,265}{12} = 7863.8 \]

Computed F statistic = \( \frac{5496.1}{7863.8} = 0.7 \)

Theoretical \( F(0.9;2,12) = 2.8 \)

Thus, we can conclude that the factor under consideration is not statistically significant.

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**Additional Reading**

- Visual Diagnostic Tests for verifying assumptions - Section 20.6
- Confidence intervals for Effects - Section 20.7
- Unequal sample sizes - Section 20.8
Two factor full factorial designs without replications

- Experiment design with two factors, each of which can have an arbitrary number of levels
  - Initially, we will not consider replications
  - Factors A and B, with number of levels \( a \) and \( b \) respectively, number of experiments is \( ab \)

Model

\[ y_{ij} = \mu + \alpha_j + \beta_i + e_{ij}, \] where \( y_{ij} \) is the observed response with the factor A at level \( j \) and the factor B at level \( i \), \( \mu \) is the mean response, \( \alpha_j \) is the effect of factor A at level \( j \), and \( \beta_i \) is the effect of factor B at level \( i \), and \( e_{ij} \) is the error term.

The effects are computed such that

\[
\sum \alpha_j = 0, \quad \sum \beta_i = 0, \quad \sum_j \sum_i e_{ij} = 0
\]

We obtain \( \overline{y} = \mu \), \( \overline{y}_i = \mu + \beta_i \), \( \overline{y}_j = \mu + \alpha_j \)
Computation of Effects

<table>
<thead>
<tr>
<th>Workload</th>
<th>Two Caches</th>
<th>One Cache</th>
<th>No Cache</th>
<th>Row Sum</th>
<th>Row Mean</th>
<th>Row Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASM</td>
<td>54.0</td>
<td>55.0</td>
<td>106.0</td>
<td>215.0</td>
<td>71.7</td>
<td>-0.5βᵢ</td>
</tr>
<tr>
<td>TECO</td>
<td>60.0</td>
<td>60.0</td>
<td>123.0</td>
<td>243.0</td>
<td>81.0</td>
<td>8.8</td>
</tr>
<tr>
<td>SIEVE</td>
<td>43.0</td>
<td>43.0</td>
<td>120.0</td>
<td>206.0</td>
<td>68.7</td>
<td>-3.5</td>
</tr>
<tr>
<td>DHRYSTONE</td>
<td>49.0</td>
<td>52.0</td>
<td>111.0</td>
<td>212.0</td>
<td>70.7</td>
<td>-1.5</td>
</tr>
<tr>
<td>SORT</td>
<td>49.0</td>
<td>50.0</td>
<td>108.0</td>
<td>207.0</td>
<td>69.0</td>
<td>-3.2</td>
</tr>
</tbody>
</table>

Column Sum          255.0 | 260.0 | 568.0 | 1083.0 |
Column Mean         51.0 | 52.0 | 113.6 | 72.2   | \(\bar{y}_{..}\) |
Column effect \(\alpha_i\) -21.2 | -20.2 | 41.4 |

Estimating experimental errors

\[ \hat{y}_j = \mu + \alpha_j + \beta_i \]
\[ e_{ij} = y_{ij} - \hat{y}_j = y_{ij} - \mu - \alpha_j - \beta_i \]
\[ SSE = \sum_{i=1}^{b} \sum_{j=1}^{a} e_{ij}^2 \]

For our example, \( \hat{y}_{11} = 72.2 - 21.2 - 0.5 = 50.5 \)
\( e_{11} = 54 - 50.5 = 3.5 \)
\[ SSE = 3.5^2 + 0.2^2 + \ldots + (\ldots) = 236.80 \]
Allocation of variation

- We can show $SSY = SS0 + SSA + SSB + SSE$
  and $SST = SSY - SS0 = SSA + SSB + SSE$
- For our example, $SSY = 91,595$; $SS0 = 78,192.6$; $SSA = 12,857.2$; $SSB = 308.4$, $SST = 13,402.41$
  $SSE = SST - SSA - SSB = 236.8$
- The percentage of variation explained by the cache is 95.9%, due to workloads is 2.3% and the unexplained variation is 1.8%

Analysis of variance (ANOVA)

- Similar to one-factor analysis
  $MSA = SSA/(a-1); MSB = SSB/(b-1)$
  $MSE = SSE/(a-1)(b-1)$
- F-ratio for factor A is $F_A = MSA/MSE$ and for factor B is $F_B = MSB/MSE$
  - If greater than theoretical F-value then factor is statistically significant
  - For our example, $F_A = 217.2$, $F_B = 2.6$, theoretical F value = 2.8, so the first factor (cache) is statistically significant
Additional Reading

- Section 21.6 - confidence intervals for effects
- Section 21.7 - multiplicative models for two factor experiments
- Section 21.8 - handling missing observations (optional)

Two-factor full factorial design with replications

$y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ij}$, where $y_{ijk}$ is the observed response in the $k$th replication of the experiment with the factor $A$ at level $j$ and the factor $B$ at level $i$, $\mu$ is the mean response, $\alpha_j$ is the effect of factor $A$ at level $j$, $\beta_i$ is the effect of factor $B$ at level $i$, $\gamma_{ij}$ is the effect of interaction between factor $A$ at level $j$ and factor $B$ at level $i$, and $e_{ij}$ is the experimental error.

The effects are computed such that

$\sum \alpha_j = 0, \quad \sum \beta_i = 0,$

The interactions are computed so that their row as well as column sums are 0

The errors in each experiment add to 0

$\sum_{k=1}^{r} e_{ijk} = 0, \quad \forall i,j$
Computation of effects

- The observations are arranged in $b$ rows and $a$ columns with each cell containing $r$ observations.
- Compute the average of the $r$ observations for each cell.
- Then, proceed as in the analysis of two-factor design without replication.

Computation of errors

$$\hat{y}_{ij} = \mu + \alpha_j + \beta_i + \gamma_{ij} = \bar{y}_{ij}. \quad \text{(the average of the $r$ observations in a cell)}$$

The error in the $k$th replication of the experiment is $e_{ijk} = y_{ijk} - \bar{y}_{ij}$.

$$SSE = \sum_{i=1}^{b} \sum_{j=1}^{a} \sum_{k=1}^{r} e_{ijk}^2$$
Allocation of variation and ANOVA

- We can show
  \[ \text{SST} = \text{SSY} - \text{SSO} = \text{SSA} + \text{SSB} + \text{SSAB} + \text{SSE} \]

- ANOVA
  - F-test: compute \( \frac{\text{MSA}}{\text{MSE}} \), \( \frac{\text{MSB}}{\text{MSE}} \), \( \frac{\text{MSAB}}{\text{MSE}} \)
  - Degrees of freedom: SSA has \( a-1 \), SSB has \( b-1 \), SSAB has \( (a-1)(b-1) \), SSE has \( ab(r-1) \)

Further Reading

- Section 22.6 - confidence intervals for effects
- Chapter 23 - General Full factorial designs with \( k \) factors
  - Generalization of analysis techniques discussed
  - Informal (non-statistical methods) for determining the important factors