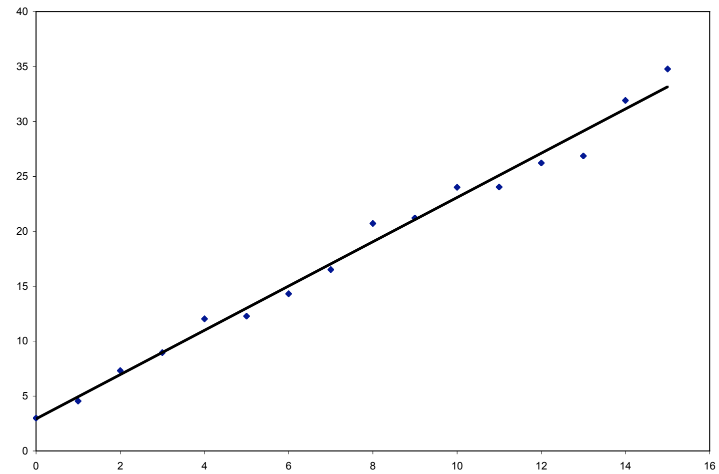
Simple Regression

CS 700

Basics

- Purpose of regression analysis: predict the value of a dependent or response variable from the values of at least one explanatory or independent variable (also called predictors or factors).
- Purpose of correlation analysis: measure the strength of the correlation between two variables.

Linear Relationship



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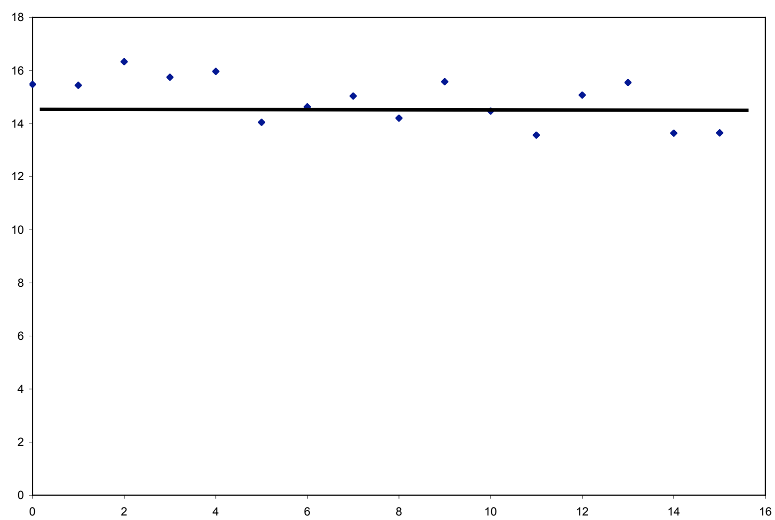
No Relationship



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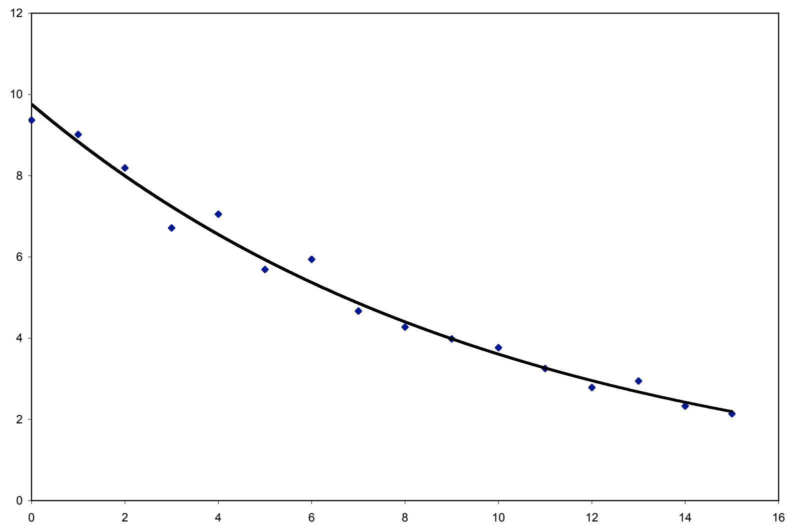
Negative Curvilinear



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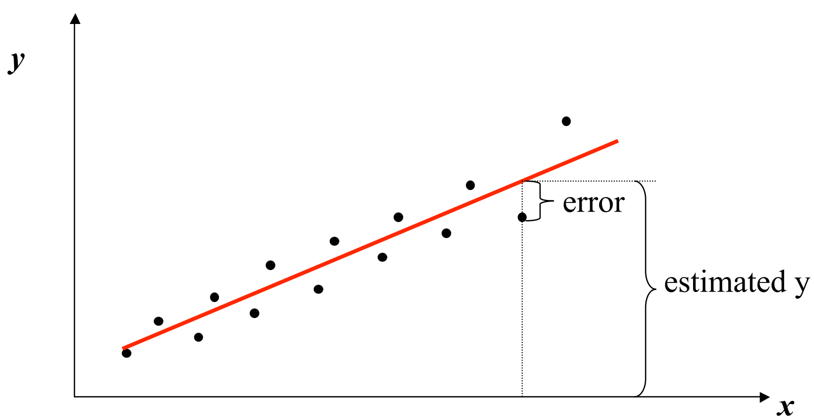
Simple Linear Regression
Residual Error



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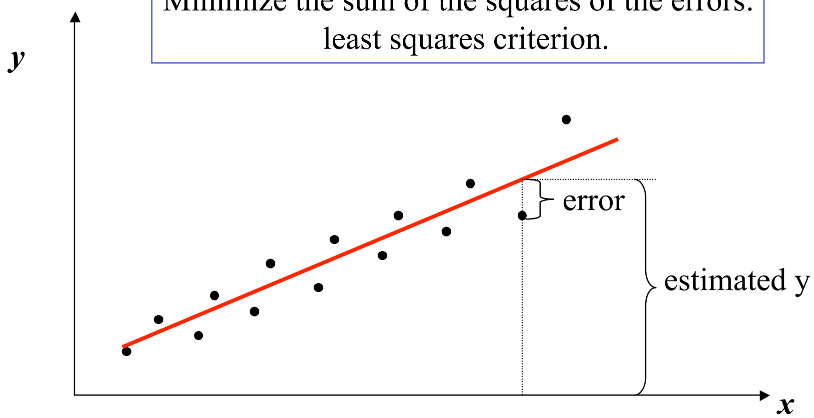
Simple Linear Regression
Selecting the "best" line

Minimize the sum of the squares of the errors: least squares criterion.



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Linear Regression

$$\hat{Y}_i = b_0 + b_1 X_i$$

$\hat{Y}_i$: predicted value of Y for observation i.

X_i : value of observation i.

b_0 and b_1 are chosen to minimize:

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} \left[Y_i - (b_0 + b_1 X_i)\right]^2$$

Subject to: $\sum_{i=1}^{n} e_i = 0$

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Method of Least Squares

$$b_1 = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{X}\overline{Y}}{\sum_{i=1}^{n} X_i^2 - n\left(\overline{X}\right)^2}$$

$$b_0 = \overline{Y} - b_1\overline{X}$$

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Linear Regression Example

Number of I/Os (x)	CPU Time (y)	Estimate (0.0408*x +0.0508)	Error	Error Squared
1	0.092	0.092	0.0005	0.00000
2	0.134	0.132	0.0013	0.00000
3	0.165	0.173	-0.0083	0.00007
4	0.211	0.214	-0.0026	0.00001
5	0.242	0.255	-0.0128	0.00016
6	0.302	0.295	0.0067	0.00005
7	0.357	0.336	0.0206	0.00042
8	0.401	0.377	0.0239	0.00057
9	0.405	0.418	-0.0131	0.00017
10	0.442	0.459	-0.0161	0.00026
				0.00171

Xbar	5.5
Ybar	0.275
Sum x2	385
Sum xy	18.494616
b1	0.0408
b0	0.0508

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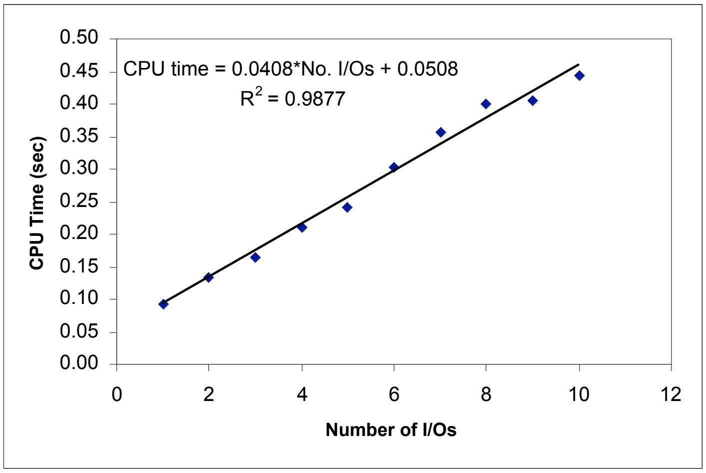
Linear Regression Example



Allocation of Variation

- No regression model: use mean as predicted value. SSE is:

$$SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$ ← Sum of squares total

$$SSR = SST - SSE$$ ← Sum of squares explained by the regression.

(SSE: Variation not explained by regression)

Allocation of Variation

- Coefficient of determination (R^2): fraction of variation explained by the regression.

$$R^2 = \frac{SSR}{SST} = \frac{SST - SSE}{SST} = 1 - \frac{SSE}{SST}$$

The closer R^2 is to one, the better is the regression model.

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Number of I/Os (x)	CPU Time (y)	Estimate (0.0408*x +0.0508)	Error	Error Squared	SSY
1	0.092	0.092	0.0005	0.00000	0.00848
2	0.134	0.132	0.0013	0.00000	0.017882
3	0.165	0.173	-0.0084	0.00007	0.027173
4	0.211	0.214	-0.0027	0.00001	0.044645
5	0.242	0.255	-0.0129	0.00017	0.058505
6	0.302	0.296	0.0066	0.00004	0.091331
7	0.357	0.336	0.0204	0.00042	0.127331
8	0.401	0.377	0.0238	0.00056	0.160771
9	0.405	0.418	-0.0133	0.00018	0.163795
10	0.442	0.459	-0.0163	0.00027	0.195783
	0.275			0.00172	0.89570

SST	0.1388841
SSR	0.1371690
R2	0.9876514

SSE SSY

$$SST = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \left(\sum_{i=1}^{n} Y_i^2\right) - n\overline{Y}^2 = SSY - SS0$$

$$SSE = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

$$SSR = \sum_{i=1}^{n}\left(\hat{Y}_i - \overline{Y}\right)^2 = SST - SSE$$

$R^2 = \frac{SSR}{SST}$ coefficient of determination.

The higher the value of R^2 the better the regression.

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Standard Deviation of Errors

- Variance of errors: divide the sum of squares (SSE) by the number of degrees of freedom (n-2 since two regression parameters need to be computed first).

$$s_e^2 = \frac{SSE}{n-2} \longleftarrow \text{Mean squared error (MSE)}$$

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Degrees of freedom of various sum of squares.

SST	n-1	Need to compute $\bar{Y}$
SSY	n	Does not depend on any other parameter
SS0	1	
SSE	n-2	Need to compute two regression parameters
SSR	1	=SST-SSE

Degrees of freedom add as sum of squares do.

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Confidence Interval for Regression Parameters

- b_o and b_1 were computed from a sample. So, they are just estimates of the true parameters β_0 and β_1 for the true model.
- Standard deviations for b_o and b_1.

$$s_{b_0} = s_e \sqrt{\frac{1}{n} + \frac{(\bar{X})^2}{\sum_{i=1}^{n} X_i^2 - n(\bar{X})^2}}$$

$$s_{b_1} = \frac{s_e}{\sqrt{\sum_{i=1}^{n} X_i^2 - n(\bar{X})^2}}$$

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Confidence Interval for Regression Parameters

100(1-α)% confidence interval for b_o and b_1

$$b_0 \pm t_{[1-\alpha/2;n-2]} s_{b_0}$$

$$b_1 \pm t_{[1-\alpha/2;n-2]} s_{b_1}$$

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Confidence Interval Example

Number of I/Os (x)	CPU Time (y)	Estimate (0.0408*x +0.0508)	Error	Error Squared
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9	0.405	0.418	-0.0131	0.00017
10	0.442	0.459	-0.0161	0.00026
			SSE:	0.00171

Xbar	5.5		
Ybar	0.275		
Sum x2	385		
Sum xy	18.494616		
b1	0.0408		
b0	0.0508		
se^2	0.0002144	Lower bo	0.027772
se	0.0146411	Upper bo	0.073900
sb0	0.0100017		
sb1	0.0016119	Lower b1	0.037058576
95% confidence level		Upper b1	0.044492804
alpha	0.05		
t[1-alpha/2;n-2]	2.3060056		
SST	0.1388841		
SSR	0.13717		
R2	0.9876524		

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Confidence Interval for the Predicted Value

- The standard deviation of the mean of a future sample of *m* observations at $X = X_p$ is

$$s_{\hat{y}_{mp}} = s_e \left[\frac{1}{m} + \frac{1}{n} + \frac{\left(X_p - \bar{X}\right)^2}{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2} \right]^{1/2}$$

As the future sample size (m) increases, the standard deviation for predicted value decreases.

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Confidence Interval for the Predicted Value

100(1-α)% confidence interval for the predicted value for a future sample of size *m* at X_p:

$$\hat{y}_p \pm t_{[1-\alpha/2;n-2]} s_{\hat{y}_{mp}}$$

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Linear Regression Assumptions

- Linear relationship between the response (y) and the predictor (x).
- The predictor (x) is non-stochastic and is measured without any error.
- Errors are statistically independent.
- Errors are normally distributed with zero mean and a constant standard deviation.

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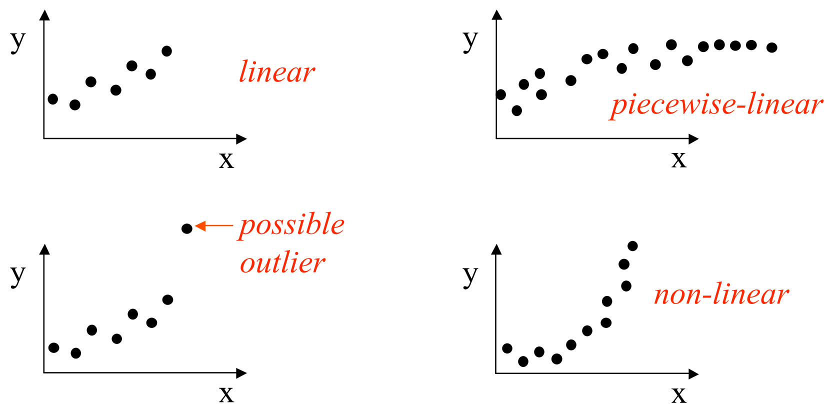
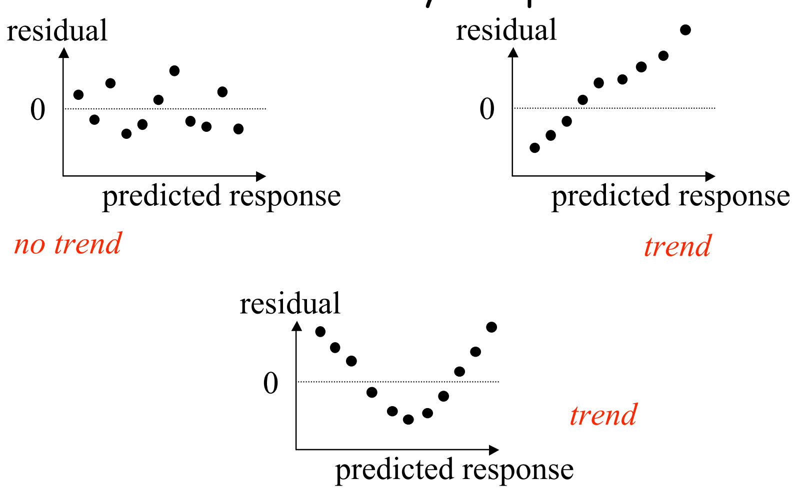
Linear Regression Assumptions

Linear relationship between the response (y) and the predictor (x).



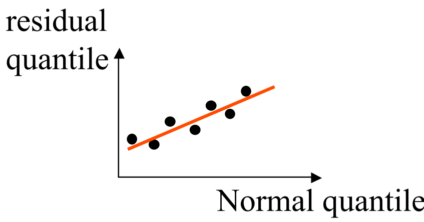
Linear Regression Assumptions

Errors are statistically independent.

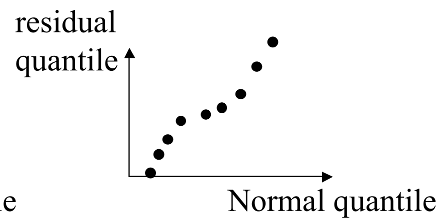


Linear Regression Assumptions

Errors are normally distributed.



normally distributed errors



non-normally distributed errors

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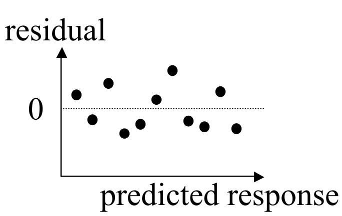
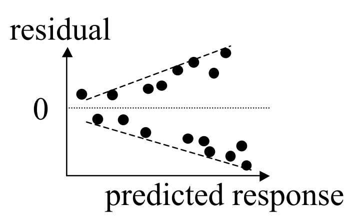
Linear Regression Assumptions

Errors have a constant standard deviation.



no trend in spread



increasing spread

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Other Regression Models

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Multiple Linear Regression

- Use to predict the value of the response variable as function of k predictor variables $x_1, \ldots, x_n$.

$$\hat{Y}_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + \ldots + b_x X_{ki}$$

- Similar to simple linear regression.
- MS Excel can be used to do multiple linear regression.

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CPU Time (yi)	I/O Time (x1i)	Memory Requirement (x2i)
2	14	70
5	16	75
7	27	144
9	42	190
10	39	210
13	50	235
20	83	400

Want to find:

$$CPUTime = b_0 + b_1 * I/OTime + b_2 * MemoryRequirement$$

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.9870 ← R
R Square	0.9742
Adjusted R Square	0.9614
Standard Error	1.1511
Observations	7

	Coefficients	*Standard Error*	*t Stat*	*Lower 95%*	*Upper 95%*	*Lower 90.0%*	*Upper 90.0%*
Intercept (b0)	-0.16145	0.91345	-0.17674	-2.69759	2.37470	-2.10878	1.78589
X Variable 1 (b1)	0.11824	0.19260	0.61389	-0.41652	0.65299	-0.29236	0.52884
X Variable 2 (b2)	0.02650	0.04045	0.65519	-0.08580	0.13881	-0.05973	0.11273

Curvilinear Regression

Approach: plot a scatter plot. If it does not look linear, try non-linear models:

Non-linear	Linear
$y = a + b / x$	$y = a + b(1/x)$
$y = 1/(a + bx)$	$(1/y) = a + bx$
$y = x/(a + bx)$	$(x/y) = a + bx$
$y = a \times b^x$	$\ln y = \ln a + x \ln b$
$y = a + bx^n$	$y = a + b(x^n)$

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