Inference in Bayesian networks

AIMA2e Chapter 14.4–5

Outline

◊ Exact inference by enumeration

◊ Exact inference by variable elimination

◊ Approximate inference by stochastic simulation

◊ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(\mathbf{X_i}|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(\mathbf{X_i}, \mathbf{X_j}|\mathbf{E}=\mathbf{e}) = \mathbf{P}(\mathbf{X_i}|\mathbf{E}=\mathbf{e})\mathbf{P}(\mathbf{X_j}|\mathbf{X_i}, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for
$P(outcome|action, evidence)$

Value of information: which evidence to seek next?

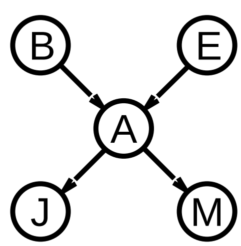
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$\mathbf{P}(\mathbf{B}|\mathbf{j}, \mathbf{m})$
$= \mathbf{P}(\mathbf{B}, \mathbf{j}, \mathbf{m})/\mathbf{P}(\mathbf{j}, \mathbf{m})$
$= \alpha \mathbf{P}(\mathbf{B}, \mathbf{j}, \mathbf{m})$
$= \alpha \sum_e \sum_a \mathbf{P}(\mathbf{B}, \mathbf{e}, \mathbf{a}, \mathbf{j}, \mathbf{m})$

Rewrite full joint entries using product of CPT entries:
$\mathbf{P}(\mathbf{B}|\mathbf{j}, \mathbf{m})$
$= \alpha \sum_e \sum_a \mathbf{P}(\mathbf{B})\mathbf{P}(\mathbf{e})\mathbf{P}(\mathbf{a}|\mathbf{B}, \mathbf{e})\mathbf{P}(\mathbf{j}|\mathbf{a})\mathbf{P}(\mathbf{m}|\mathbf{a})$
$= \alpha \mathbf{P}(\mathbf{B}) \sum_{\mathbf{e}} \mathbf{P}(\mathbf{e}) \sum_{\mathbf{a}} \mathbf{P}(\mathbf{a}|\mathbf{B}, \mathbf{e})\mathbf{P}(\mathbf{j}|\mathbf{a})\mathbf{P}(\mathbf{m}|\mathbf{a})$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

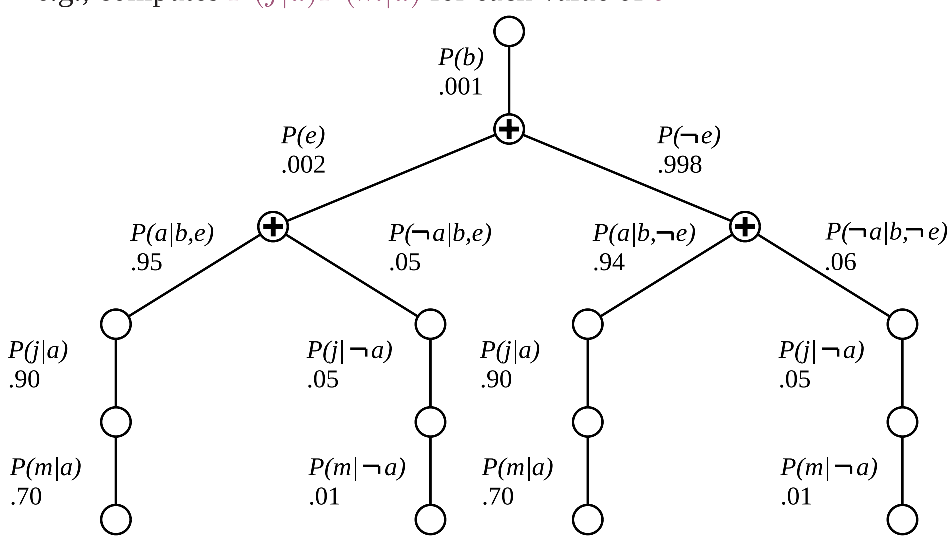
```
function ENUMERATION-ASK(X, e, bn) returns a distribution over X
  inputs: X, the query variable
          e, observed values for variables E
          bn, a Bayesian network with variables {X} ∪ E ∪ Y

  Q(X) ← a distribution over X, initially empty
  for each value x_i of X do
     extend e with value x_i for X
     Q(x_i) ← ENUMERATE-ALL(VARS[bn], e)
  return NORMALIZE(Q(X))

function ENUMERATE-ALL(vars, e) returns a real number
  if EMPTY?(vars) then return 1.0
  Y ← FIRST(vars)
  if Y has value y in e
     then return P(y | Pa(Y)) × ENUMERATE-ALL(REST(vars), e)
     else return Σ_y P(y | Pa(Y)) × ENUMERATE-ALL(REST(vars), e_y)
          where e_y is e extended with Y = y
```


Evaluation tree

Enumeration is inefficient: repeated computation
e.g., computes $P(j|a)P(m|a)$ for each value of e



Inference by variable elimination

Variable elimination: carry out summations right-to-left, storing intermediate results (factors) to avoid recomputation

$$
\begin{aligned}
&\mathbf{P}(\mathbf{B}|\mathbf{j},\mathbf{m})\\
&= \alpha \underbrace{\mathbf{P}(\mathbf{B})}_{B} \sum_e \underbrace{P(e)}_{E} \sum_a \underbrace{\mathbf{P}(\mathbf{a}|\mathbf{B},\mathbf{e})}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}\\
&= \alpha \mathbf{P}(\mathbf{B}) \textstyle\sum_{\mathbf{e}} \mathbf{P}(\mathbf{e}) \sum_{\mathbf{a}} \mathbf{P}(\mathbf{a}|\mathbf{B},\mathbf{e}) \mathbf{P}(\mathbf{j}|\mathbf{a}) \mathbf{f}_{\mathbf{M}}(\mathbf{a})\\
&= \alpha \mathbf{P}(\mathbf{B}) \textstyle\sum_{\mathbf{e}} \mathbf{P}(\mathbf{e}) \sum_{\mathbf{a}} \mathbf{P}(\mathbf{a}|\mathbf{B},\mathbf{e}) \mathbf{f}_{\mathbf{J}}(\mathbf{a}) \mathbf{f}_{\mathbf{M}}(\mathbf{a})\\
&= \alpha \mathbf{P}(\mathbf{B}) \textstyle\sum_{\mathbf{e}} \mathbf{P}(\mathbf{e}) \sum_{\mathbf{a}} \mathbf{f}_{\mathbf{A}}(\mathbf{a},\mathbf{b},\mathbf{e}) \mathbf{f}_{\mathbf{J}}(\mathbf{a}) \mathbf{f}_{\mathbf{M}}(\mathbf{a})\\
&= \alpha \mathbf{P}(\mathbf{B}) \textstyle\sum_{\mathbf{e}} \mathbf{P}(\mathbf{e}) \mathbf{f}_{\bar{\mathbf{A}}\mathbf{JM}}(\mathbf{b},\mathbf{e}) \text{ (sum out } A\text{)}\\
&= \alpha \mathbf{P}(\mathbf{B}) \mathbf{f}_{\bar{\mathbf{E}}\bar{\mathbf{A}}\mathbf{JM}}(\mathbf{b}) \text{ (sum out } E\text{)}\\
&= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:
move any constant factors outside the summation
add up submatrices in pointwise product of remaining factors

$$\textstyle\sum_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \sum_x f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

$$f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l) = f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l)$$

E.g., $f_1(a,b) \times f_2(b,c) = f(a,b,c)$

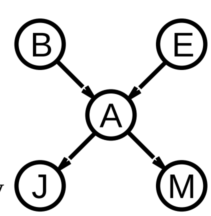
Variable elimination algorithm

function ELIMINATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, evidence specified as an event
    bn, a belief network specifying joint distribution $\mathbf{P}(\mathbf{X_1},\ldots,\mathbf{X_n})$

  $factors \leftarrow [\,]$; $vars \leftarrow$ REVERSE(VARS[bn])
  for each var **in** $vars$ **do**
    $factors \leftarrow$ [MAKE-FACTOR(var, **e**)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Irrelevant variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over m is identically 1; M is *irrelevant* to the query

Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so M is irrelevant

(Compare this to backward chaining from the query in Horn clause KBs)

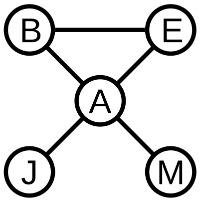
Irrelevant variables contd.

Defn: moral graph of Bayes net: marry all parents and drop arrows

Defn: **A** is m-separated from **B** by **C** iff separated by **C** in the moral graph

Thm 2: Y is irrelevant if m-separated from X by **E**

For $P(JohnCalls|Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant

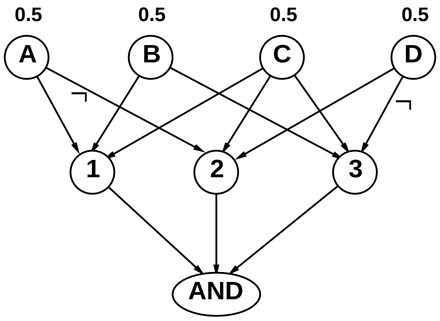


Complexity of exact inference

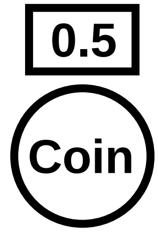
Singly connected networks (or polytrees):
– any two nodes are connected by at most one (undirected) path
– time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:
– can reduce 3SAT to exact inference $\Rightarrow$ NP-hard
– equivalent to *counting* 3SAT models $\Rightarrow$ #P-complete

1. A v B v C
2. C v D v ¬A
3. B v C v ¬D



Inference by stochastic simulation



Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P

Outline:

– Sampling from an empty network
– Rejection sampling: reject samples disagreeing with evidence
– Likelihood weighting: use evidence to weight samples
– Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

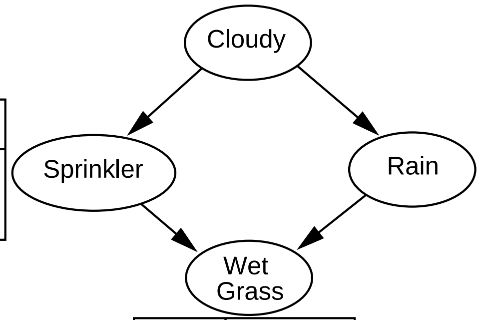
Sampling from an empty network

function PRIOR-SAMPLE(*bn*) **returns** an event sampled from *bn*
inputs: *bn*, a belief network specifying joint distribution $\mathbf{P}(\mathbf{X_1}, \ldots, \mathbf{X_n})$

x ← an event with n elements
for $i = 1$ **to** n **do**
x_i ← a random sample from $\mathbf{P}(\mathbf{X_i} \mid \mathbf{Parents}(\mathbf{X_i}))$
return x

Example

P(C)
.50



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T	.10
F	.50

C	P(R|C)
T	.80
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S	R	P(W|S,R)
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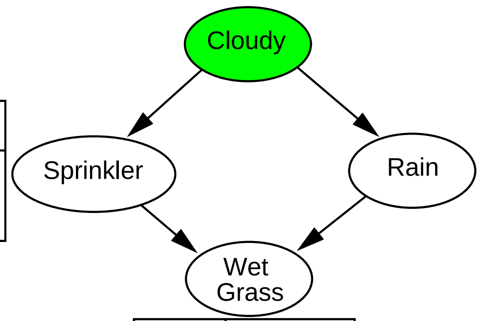

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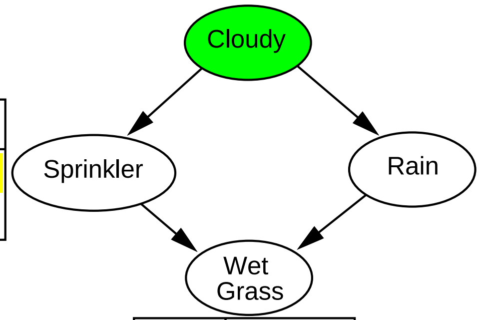

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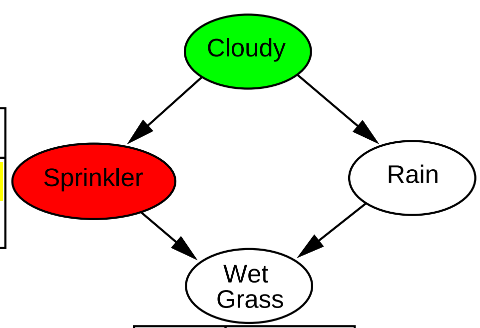

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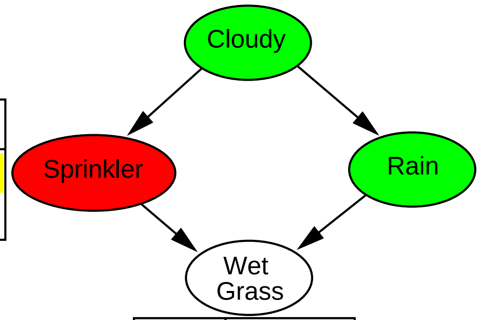

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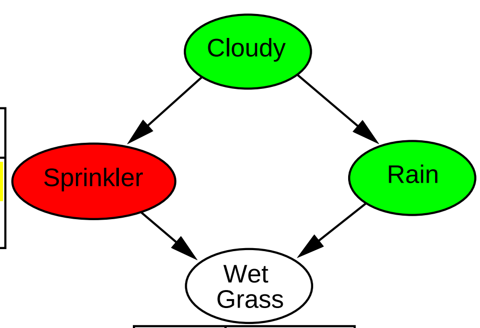

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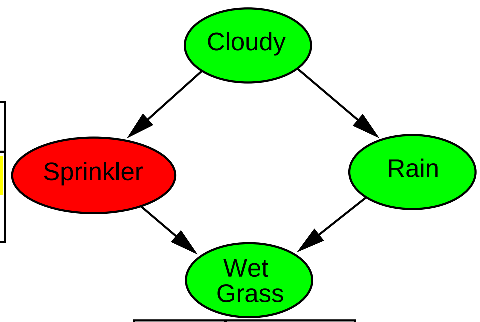

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Sampling from an empty network contd.

Probability that PRIORSAMPLE generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|Parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t,f,t,t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t,f,t,t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N\to\infty} \hat{P}(x_1,\ldots,x_n) &= \lim_{N\to\infty} N_{PS}(x_1,\ldots,x_n)/N \\ &= S_{PS}(x_1,\ldots,x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1,\ldots,x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function REJECTION-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
  local variables: N, a vector of counts over X, initially zero

  for j = 1 to N do
     x ← PRIOR-SAMPLE(bn)
     if x is consistent with e then
        N[x] ← N[x]+1 where x is the value of X in x
  return NORMALIZE(N[X])
```

E.g., estimate $\mathbf{P}(\mathbf{Rain}|\mathbf{Sprinkler} = \mathbf{true})$ using 100 samples

27 samples have $Sprinkler = true$

Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(< 8, 19 >) =< 0.296, 0.704 >$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X, \mathbf{e})$ (algorithm defn.)

$= \mathbf{N}_{PS}(X, \mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)

$\approx \mathbf{P}(\mathbf{X}, \mathbf{e})/\mathbf{P}(\mathbf{e})$ (property of PRIORSAMPLE)

$= \mathbf{P}(\mathbf{X}|\mathbf{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

```
function LIKELIHOOD-WEIGHTING(X, e, bn, N) returns an estimate of P(X|e)
   local variables: W, a vector of weighted counts over X, initially zero

   for j = 1 to N do
      x, w ← WEIGHTED-SAMPLE(bn)
      W[x] ← W[x] + w where x is the value of X in x
   return NORMALIZE(W[X])

function WEIGHTED-SAMPLE(bn, e) returns an event and a weight

   x ← an event with n elements; w ← 1
   for i = 1 to n do
      if X_i has a value x_i in e
         then w ← w × P(X_i = x_i | Parents(X_i))
         else x_i ← a random sample from P(X_i | Parents(X_i))
   return x, w
```


Likelihood weighting example

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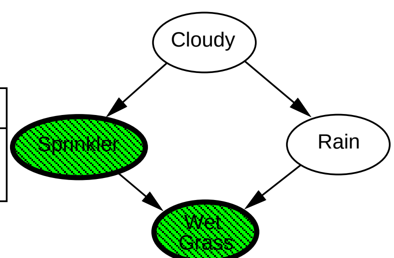

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$w = 1.0$

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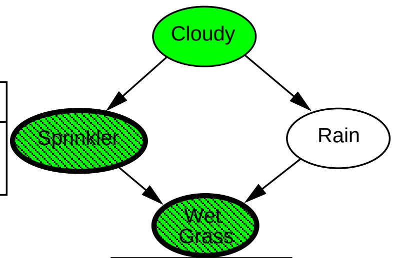

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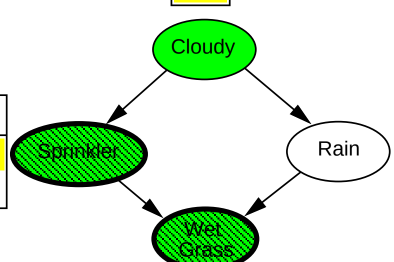

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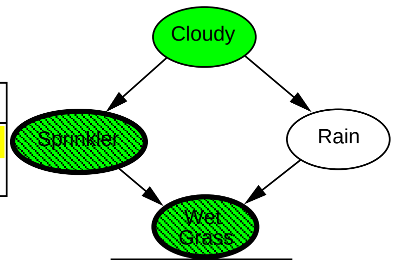

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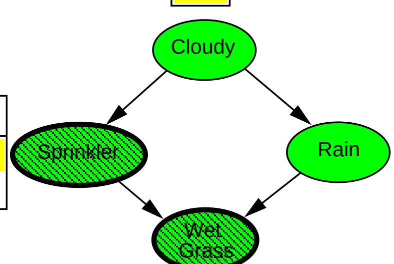

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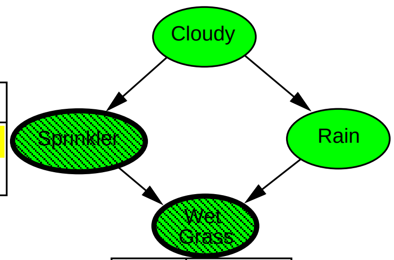

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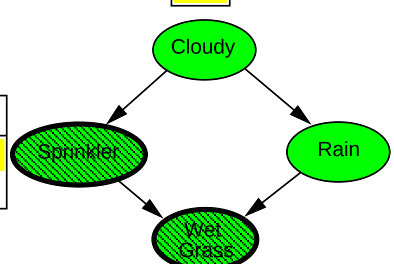
Likelihood weighting example



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$w = 1.0 \times 0.1 \times 0.99 = 0.099$

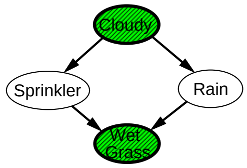
Likelihood weighting analysis

Sampling probability for WEIGHTEDSAMPLE is

$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i|Parents(Z_i))$

Note: pays attention to evidence in ancestors only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i|Parents(E_i))$

Weighted sampling probability is

$S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e})$

$= \prod_{i=1}^{l} P(z_i|Parents(Z_i)) \; \prod_{i=1}^{m} P(e_i|Parents(E_i))$

$= P(\mathbf{z}, \mathbf{e})$ (by standard global semantics of network)

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

Approximate inference using MCMC

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

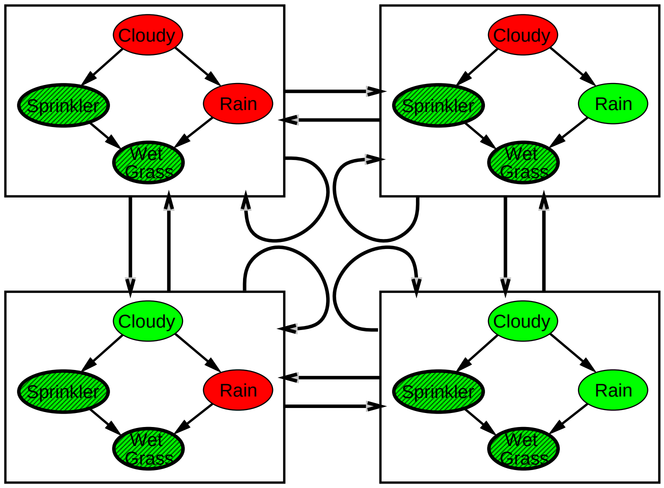
function MCMC-ASK(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

  initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
  for j = 1 to N **do**
    $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{x}$ from $\mathbf{P}(\mathbf{Z_i}|\mathbf{MB}(\mathbf{Z_i}))$ given the values of $MB(Z_i)$ in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

Can also choose a variable to sample at random each time

The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(\mathbf{Rain}|\mathbf{Sprinkler} = \mathbf{true}, \mathbf{WetGrass} = \mathbf{true})$

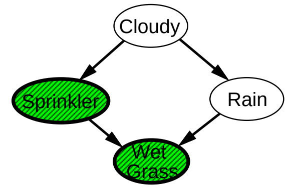
Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

E.g., visit 100 states
31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
$= \text{NORMALIZE}(< 31, 69 >) =< 0.31, 0.69 >$

Theorem: chain approaches stationary distribution:
long-run fraction of time spent in each state is exactly
proportional to its posterior probability

Markov blanket sampling



Markov blanket of $Cloudy$ is

$Sprinkler$ and $Rain$

Markov blanket of $Rain$ is

$Cloudy$, $Sprinkler$, and $WetGrass$

Probability given the Markov blanket is calculated as follows:

$P(x_i'|MB(X_i)) =$
$P(x_i'|Parents(X_i)) \prod_{Z_j \in Children(X_i)} P(z_j|Parents(Z_j))$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

1) Difficult to tell if convergence has been achieved

2) Can be wasteful if Markov blanket is large:

$P(X_i|MB(X_i))$ won't change much (law of large numbers)

Summary

Exact inference by variable elimination:

– polytime on polytrees, NP-hard on general graphs

– space = time, very sensitive to topology

Approximate inference by LW, MCMC:

– LW does poorly when there is lots of (downstream) evidence

– LW, MCMC generally insensitive to topology

– Convergence can be very slow with probabilities close to 1 or 0

– Can handle arbitrary combinations of discrete and continuous variables